Mathematics for Homeschooling

Michael Gannotti

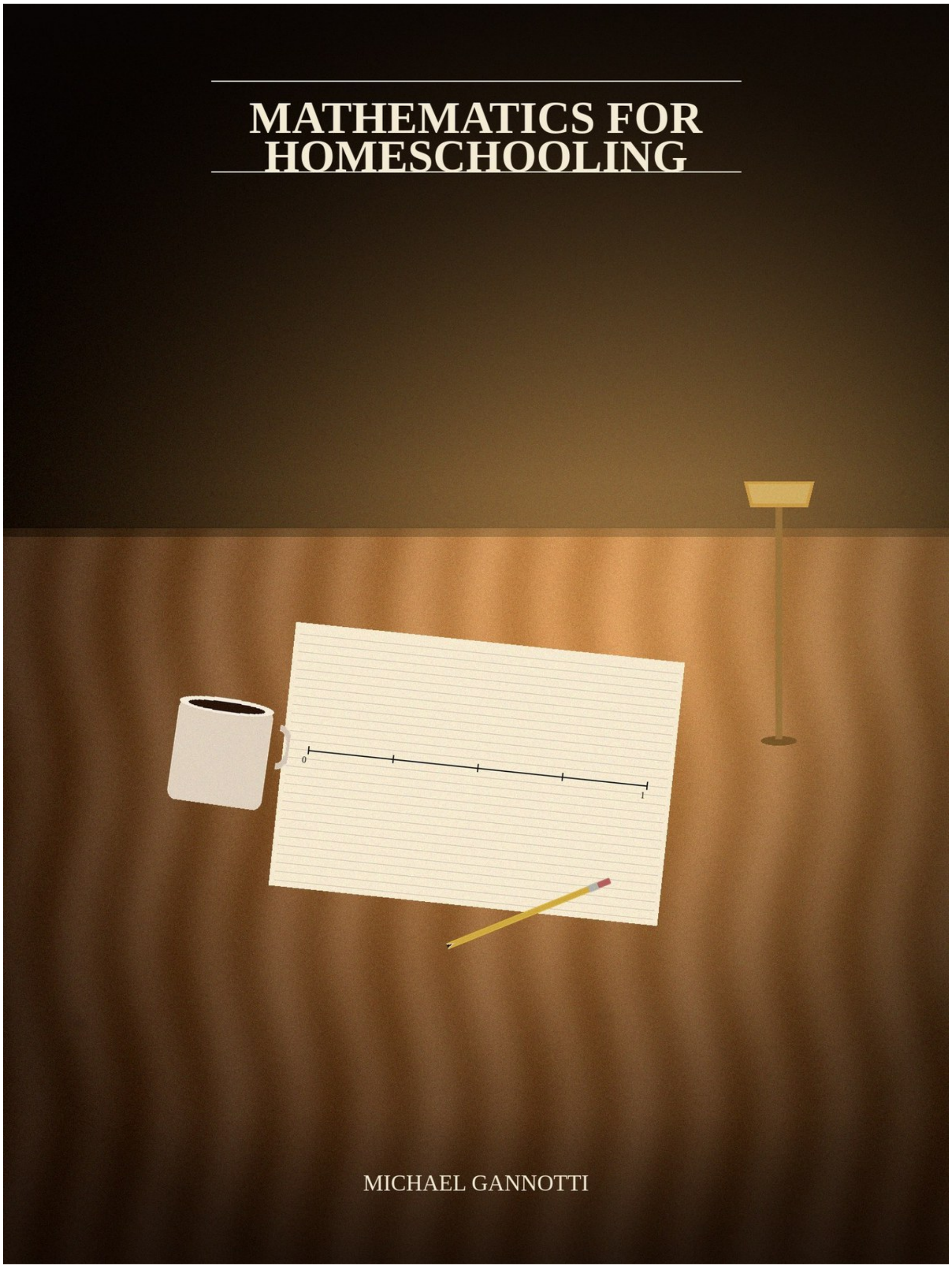

Figure 1: Cover

Mathematics for Homeschooling

Michael Gannotti

This illustrated edition is the teaching-manual of 30 August 2026. Endnotes run in one series. Chapter-opening images are original still-lifes, not portraits of persons or children. Documents still constrain the facts.

Contents

Welcome

This book exists because the mathematics hour at your table is the whole program. There is no department down the hall and no specialist waiting after lunch. There is you, a student, and today's idea. That is enough, if you know what to do with the hour.

I wrote this for a capable, busy, willing parent. You may be teaching two ages at once. You may be fitting math between a job, a toddler, and a grocery run. You may have loved school mathematics, or you may remember it as a fog. Many adults feel rusty when they sit down to teach. That feeling is common. It is not a verdict. We will move on from it. This book will make you fluent enough in today's idea to notice a wrong turn and ask a good question — without taking over the pencil.

You do not need to be a mathematician. You do need to understand this week's idea well enough to hear (1/2 + 1/3 = 2/5) as a size mistake, not a cute slip. You need a session shape you can run on a Tuesday. You need a few sentences that actually help. That is the job. A path through mathematics — from first-grade number sense through high-school calculus — is possible at a kitchen table. The path is not a personality trait and it is not a percentile. It is a sequence of ideas, practiced until they hold, with you in the chair.

About 3.4 percent of U.S. students ages 5–17 were homeschooled in 2022–23 — roughly 1.765 million children.[1] That figure is context, not a ranking. You are one of those tables. The hour in front of you is the work.

[1] Ellena Sempeles and Jiashan Cui, *Parent and Family Involvement in Education: 2023*, NCES 2024-113 (Washington, DC: National Center for Education Statistics, September 2024), Table A-6: 3.4 percent homeschooled, approximately 1,765,000 students, ages 5–17 with a K–12 grade equivalent, 2022–23. https://nces.ed.gov/pubs2024/2024113.pdf. Access date for URLs in these notes: 30 August 2026. Latest federal count this book uses; not a 2026 national total.

What a good math hour looks like

You sit down already knowing today's idea. The child warms up on facts they can already get right. You model one problem out loud, short. Then they try, and you wait. You ask one good question — not "did you get it?" but "how did you know?" Practice mixes the new move with last week's. The hour ends with one or two problems they do alone, book closed, no hints. You stop talking sooner than feels polite. The child holds the pencil. You hold the key. That is the hour. The next long piece of this front matter, *The Math Hour*, will teach it in full. Later chapters will not reinvent it.

What you will actually get

Each teaching chapter does eight jobs, always in the same order, so you are never hunting for the move.

You will learn why this week's idea is worth the struggle — what it unlocks later. You will understand the idea yourself, in plain language, with one everyday picture and one precise picture, and with the wrong answers you should be able to hear. You will get a session you can run this week, including exact wording. You will get practice that builds learning, not a pile of the same problem. Your student will get a short section of their own. If it isn't clicking, you will get three diagnostics and a next move for each, with no shame in the room. Tools, including AI, stay optional and adult-side. And you will get a plain checklist for "done enough," so you can place by skill rather than by birthday.

This week you can learn the session shape and today's idea well enough to hear a wrong answer. Today the child can try one problem without you holding the pencil.

What this book will not do

This book will not hand you 180 days of worksheets. A year of photocopies is not a teaching method, and I will not pretend it is.

It will not sell you a curriculum. Later, a short resources chapter names common programs by fit — parent load, style, how they place a student — so you can choose.

Saxon, Singapore, Math-U-See, Beast Academy, Teaching Textbooks, and the rest are tools. None of them is this book, and this book is not a catalog in disguise.

It will not promise a score. There is no guaranteed percentile, no guaranteed SAT number, and no guaranteed college letter inside these pages. What this book can promise is a path: the ideas in order, at the skill the child actually has, until they can do the next idea unaided.

It is not a Common Core tract, for or against. Standards are a map you can steal clusters from. They are not a homeschool mandate and they are not a method. A publisher's "grade 6" book is a scope, not a legal grade. Algebra I is a body of skill, not a tenth-birthday present.

And this is not a book that lectures your child about you, or you about your character. The student is a person, not a percentile. When this book speaks to them, it speaks with respect. No baby-talk. No research sermon.

The promise

If you only remember one sentence, remember this: a path, not a percentile.

The work is to take this child through the mathematics — counting and the equal sign, fractions on a line, ratio and signed numbers, algebra, geometry, the functions that calculus will need, data and chance — without skipping the gate because they are "good at math," and without parking them in a book that is too easy because a catalog printed a grade on the cover. You will sometimes slow down. You will sometimes skip ahead. Both are teaching. Birthday is not placement.

You can do this. You do not have to know next year's idea today. You have to know this week's idea well enough to sit still while they struggle, then ask one good question. Start here. Read *How to Use This Book This Week*, then the one-page list of five things, then *The Math Hour*. After that, open the chapter that matches the skill in front of you. You will know more after one chapter than you know this morning. Your child will have something to try today. We can do this.

How to Use This Book This Week

Start at the skill in front of you, not on page one of Grade 1 because a catalog, a birthday, or a well-meant relative said so.

This book is a path, not a calendar. The teaching chapters run: grades 1–2 (number, place value, the equal sign); grades 3–5 (multiplicative reasoning and fractions as numbers on a line); grades 6–8 (ratio, signed numbers, expressions); Algebra I; Geometry; Algebra II and the functions precalculus needs; Calculus; then data and chance. Records and resources come last. Grade labels here are sequence anchors. They tell you where an idea usually sits. They do not tell you where *this* child sits.

Pick the chapter by skill, not by birthday

Open the chapter you think is right. Skip to **What "done enough" looks like** at the end of the *previous* chapter, or to **If it isn't clicking** in the one you opened. If the child can already do those checks unaided, you are too early. If the checks from two chapters back are still failing, drop back. "Pushing through Algebra I" with a hole in fractions is how algebra becomes a fog.

If they cannot treat a fraction as a number on a line — if (1/2 + 1/3 = 2/5) still looks reasonable — stay in grades 3–5, even if they are twelve. Fractions are the elementary gate to algebra. Skipping them because a child is "good at math" is the expensive shortcut.

If they can place fractions, compute with them, and treat the equal sign as "the same amount," and they are still doing arithmetic only because the cover says Grade 7, skip ahead. What a child of a given age can learn is mostly what they have already had a chance to learn, not a lock on a birthday.[2] Homeschool tables already stretch: in the last federal subject tables that published the row, Algebra I was being taught that year to 41 percent of homeschooled students in the 6–8 band, and arithmetic was still being taught to 31 percent of the 9–12 band.[3] Both can be right. Neither is a law about your kitchen.

A publisher's "grade 6" book is a scope, not a legal grade. A ten-year-old can sit in three different "grades" in three programs on the same morning. Use this book's checklists, or a placement test from the program you own, and then teach.

Two children may need two chapters. That is ordinary. The session shape in *The Math Hour* still holds; the idea on the table changes.

How you use the parent half

Most of each teaching chapter is for you. Read it *before* the lesson, not over the child's shoulder.

Why this matters tells you what this idea unlocks. You will teach with more patience when you can see the later door.

For the parent: understand it yourself gives one everyday picture, one precise picture, and three to five wrong answers you should be able to hear. Sit with those. If (8 = 3 + 5) still looks "backwards," that is the equal-sign idea, and it is worth five quiet minutes now.

How to teach it this week assumes the session shape from *The Math Hour*. It will not rebuild the hour. It will give you the first problem, the wording for *this* idea, and how to fade your help.

[2] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Finding 15: what is developmentally appropriate is "largely contingent on prior opportunities to learn"; age-lock claims have "consistently been shown to be wrong" when prerequisite knowledge is present. ERIC ED500486, https://files.eric.ed.gov/fulltext/ED500486.pdf. Access date for URLs in these notes: 30 August 2026.

[3] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey*, NCES 2020-001 (Washington, DC: NCES, 2019), Table 9: Algebra I taught that year to 41 percent of the homeschooled 6–8 band; arithmetic still taught to 31 percent of 9–12. https://nces.ed.gov/pubs2020/2020001.pdf. Grade labels in this book are sequence anchors, not prison bars. The 2023 First Look did not republish these subject tables.

Practice that actually builds learning is what goes on the page after the model: a short blocked set of the new move, then mixed problems so the child has to choose, plus a little retrieval of facts they already know. Kitchen, money, and making can motivate or give a second picture. They do not replace the number line, the written equation, or the unaided problem.

Tools, including AI is optional, short, and for the adult. The rules live once in *The Math Hour*.

What "done enough" looks like is how you leave. You are looking for unaided work, not a perfect Tuesday.

You do not have to read the whole chapter tonight. You do have to read the parent half of *this week's* idea before you sit down with the child.

The five-minute parent warm-up

Make this a habit.

Five minutes. Child not yet in the chair. Phone face down.

1. Read today's idea until you can say it in one sentence.
2. Work two problems yourself, on paper, without the key until you finish.
3. Glance at the "wrong answers you should be able to hear." Name the one you would have given at fourteen.
4. Write one sentence you will actually say. Not a speech. Example: "The equal sign means the same amount on both sides." Or: "Show me where (3/4) lives on the line."
5. Close the book to the student page. You are ready.

That five minutes is how you stay out of their working memory. If you are learning the idea *while* they are stuck, you will talk too much or take the pencil. Prepare first. Then sit still. You need today's sentence, not a seminar.

How the student uses "For the student"

Every teaching chapter includes a short section written to the student, not about them. One or two pages. Warmer. Direct. What the idea is, a tiny worked example, two tries, an "explain it back" prompt, and one challenge problem.

Hand them that section when it is time for their attempt — after your short model, not instead of it. They may read it silently or out loud. You stay in the room. You do not add a lecture about high school.

The two tries are theirs. "Explain it back" checks that they have a picture, not only a copied step. The challenge is optional that day; it is there so a ready student does not sit in busywork.

The student page is not something to send off with an unsupervised chatbot. The attempt is still theirs. You still hold the key.

When to skip ahead

Skip ahead when this chapter's "done enough" checklist is already true *unaided*. Lingering past that is not thoroughness. It is boredom wearing a virtue.

Slow down when the same wrong answer repeats after a clear model and a real attempt. **If it isn't clicking** will give you three likely causes and a next move: a missing prerequisite, a picture that never got drawn, or a procedure with no "why." Missing facts are a fluency problem, not a character problem. Ten minutes of struggle on a well-posed new idea is ordinary work.

Diagnose fractions, signed numbers, and a relational equal sign before you buy Algebra I. Accelerate a child who has the prerequisites. Hold still for a child who does not. Both are the same kindness.

If you need a human tutor, that is a normal high-school plan, not a failure of the kitchen.

For this week: pick the chapter by skill. Do the five-minute warm-up. Run the hour as *The Math Hour* describes it. Let the student page be theirs. Stop talking sooner than you want to. That is how you use the book.

If You Only Remember Five Things

Keep this page. The chapters will add wording, pictures, and practice. They will not replace these.

1. Understand and practice together. A picture with no practice does not hold. A procedure with no picture does not transfer. When a new idea arrives, give a short "why," then a handful of problems, then return to the idea. Do not wait weeks to practice. Do not run a new algorithm with no meaning and no "why." Understanding and skill grow by taking turns, not by choosing only one.[4]

2. Fractions as numbers on a line are the gate to algebra. Pies introduce sharing. They do not, by themselves, teach that (3/4) is a number with a size. Put fractions on a line early, next to whole numbers. If (1/2 + 1/3 = 2/5) appears, that is a size mistake: the child is adding the pieces they see instead of locating a number. Stay here until the line is real. Algebra will wait. Algebra will not work around a missing gate.[5]

3. The equal sign means the same amount. It does not mean "here comes the

[4]Bethany Rittle-Johnson, Michael Schneider, and Jon R. Star, "Not a One-Way Street: Bidirectional Relations Between Procedural and Conceptual Knowledge of Mathematics," *Educational Psychology Review* 27, no. 4 (2015): 587–97. National Research Council, *Adding It Up: Helping Children Learn Mathematics*, ed. Jeremy Kilpatrick, Jane Swafford, and Bradford Findell (Washington, DC: National Academy Press, 2001), 116: the strands of proficiency are "interwoven and interdependent." Access date for URLs in these notes: 30 August 2026.

[5]National Mathematics Advisory Panel, *Foundations for Success* (Washington, DC: U.S. Department of Education, 2008): knowledge of fractions is "the most important foundational skill not currently developed among American students." Robert S. Siegler et al., "Early Predictors of High School Mathematics Achievement," *Psychological Science* 23, no. 7 (2012): 691–97: elementary fraction (and division) knowledge uniquely predicted high-school algebra five to six years later — a prediction, not a curriculum experiment. Robert Siegler et al., *Developing Effective Fractions Instruction for Kindergarten through 8th Grade*, NCEE 2010-4039 (IES, 2010), Recommendation 2: treat fractions as numbers; use the number line as a central tool.

answer." Write (8 = 3 + 5) early, and (3 + 5 = 2 + 6), so both sides can be a number. A child who always needs the blank on the right will later write (3x + 2 = 5) as "do something to 3x." Catch it in first grade if you can. Catch it before Algebra I if you must.[6]

4. Struggle before rescue. Don't grab the pencil. Ask. Wait. Hint. Then model. Ten minutes of honest work on a problem they have the pieces for is the lesson. Taking the pencil feels kind. It teaches them to watch. If they are missing facts, that is not struggle — that is a hole, and you fill the hole. If they have the pieces, sit on your hands.

5. The child attempts first. You hold the key. AI is optional for the adult. The student does the problem. You keep the answers. A tool may explain today's idea *to you*, write extra practice with the key hidden, draft a hint script, or help you diagnose work already done. It may not sit in the chair during the attempt, and it may not paste the worksheet. The full rules are in *The Math Hour*, once. Later chapters will point back rather than preach.

If this week needs a compass, this is it. Five things. Then sit down and teach.

[6] McNeil, Fyfe, Petersen, Dunwiddie, and Brletic-Shipley (2011), and McNeil, Fyfe, and Dunwiddie (2014): practicing non-canonical forms such as 4 = 2 + 2 improved understanding of equivalence. Knuth, Stephens, McNeil, and Alibali (2006): a relational view of the equal sign is connected to later algebra.

The Math Hour

The hour has a shape. Learn it once. Later chapters will give you today's idea, today's first problem, and today's sentence. They will not rebuild this hour. When a chapter says "run the session," it means this.

You do not need a school bell. You need a beginning, a short model, a real attempt, one good question, mixed practice, and an unaided check. Younger children may finish in twenty-five minutes. Older students may need forty-five. The shape does not change. The idea on the table does.

Sit down having already done the five-minute parent warm-up from *How to Use This Book This Week*. You know today's sentence. The child has the pencil. You have the key.

1. Warm-up

Three to five minutes. Facts and moves they can already get right. Mixed, not a page of the same item. This is retrieval, not a test of character.

Say:

"We're going to start with things you already know."

"Tell me two ways to make 10."

"What's 7×8? Take your time."

If they already get these right untimed, you may use a brief timer. Say:

"We'll time this only because you already get these right. Ready?"

If they do not already get them right, skip the timer. Time only what they can already produce. Fluency is quick, flexible, accurate recall — not a race and not optional. You are building retrieval, not an identity.

Keep this short. The warm-up is not the lesson. When you feel yourself teaching a new idea here, stop. That idea belongs in the model.

2. Short model

Five to eight minutes. One fully worked example, out loud. Then a second example with holes they fill. Then you stop.

New, dense material — first unlike-denominator addition, first two-step equation, first formal proof — is where a short model earns its keep. A novice staring at a blank page spends working memory searching, not learning. Show the move. Then fade.[7]

Say:

"Watch this one. I'll say every step out loud."

"The equal sign means the same amount on both sides. That's why I can write $8 = 3 + 5$."

"What does this 4 mean?"

Then, on the second example:

"Your turn to tell me what I write next. I'll hold the pencil this time. You do the thinking."

"I'm leaving the last step blank. You finish it."

Include one incorrect example when the idea is new, and ask:

"What did they think this symbol meant?"

That question is worth more than another correct copy.

[7] John Sweller and Graham Cooper, "The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra," *Cognition and Instruction* 2, no. 1 (1985): 59–89: for novices, a worked solution outperforms unguided search. Alexander Renkl and Richard K. Atkinson developed guidance fading (full example, then completion problems, then conventional problems). Once the learner can execute the procedure, extra full examples become redundant.

When they can already execute the move, stop showing full examples. Extra modeling after they can do it is noise. Fade: full example, completion problem, conventional problem. Then they work.

The model is short on purpose. If you are still talking at minute twelve, you are giving a lecture. Close the model. Hand them the first problem.

3. Student attempt

This is the center of the hour. Eight to twelve minutes. Maybe longer for a proof or a multi-step equation. The first problem is theirs.

Hand them “For the student” if the chapter has it, or the first problem from **How to teach it this week**. Then you talk less than you want to.

Say:

“This one is yours. I’ll be quiet.”

Then be quiet.

If they stall, use this order: ask, wait, hint, then model. Not the reverse.

Ask:

“Show me what you tried.”

“What do you already know about this?”

“Is that the same amount, or a different amount?”

Wait. Count a slow ten in your head. The silence is the work. If you fill it, you took the problem back.

Hint, one hint:

“Try the number line. Where would this number live?”

“Look only at this side first.”

“You already know a fact that would help. Which one?”

Then, if they are still stuck after a real try:

“I’m going to show you this one step. Then you take it from here.”

Show the step. Return the pencil.

If they are missing a fact — they cannot make 10, they cannot place 1/2 — that is a hole, not useful struggle. Fill the hole with a short retrieval or a picture, then return to the problem. Struggle on a well-posed new idea, with the pieces in place, is the lesson. Struggle on missing prerequisites is just being lost.

Keep your hands off their pencil. Taking it feels kind. It teaches them to watch. If you feel the urge, put your hands in your lap and say the ask-sentence again.

4. One good question

Two or three minutes. Not "did you get it?" They will say yes. You will learn nothing.

A good question makes them generate the idea again, in words or in a second representation.

Say:

"How did you know?"

"Can you show me a second way?"

"What would happen if this were 1/2 instead of 1/3?"

"Explain it so I could do the next one."

"Where does this number live on the line?"

One question. Maybe a follow-up. Then stop. This is not a seminar. You are checking that they have a picture, not collecting a speech.

If they cannot explain it back, they copied a step. Return to a picture — the line, the array, the written equation — and give one more attempt. Copying is what a stuck novice does when the model was too fast or the attempt was too short. Slow the model tomorrow. Keep today's pencil theirs.

5. Mixed practice

Ten to fifteen minutes. First a short blocked set of the new move — four to six problems that use today's idea — so they learn to execute. Then mix. Yesterday's type. Last month's type. A word problem of a known kind. The new move sitting next to an old one so they have to choose.

Blocked practice teaches execution. Mixed practice teaches *when* to use the move. A page of twenty identical problems feels like mastery at 9 p.m. It often is not still there in two weeks.[8]

Say:

"These first few are today's move. Get the feel of it."

Then:

"These are mixed on purpose. First decide what kind of problem it is. Then choose."

"You already know this type from last week. Find it among the new ones."

Word problems are types — change, compare, equal groups, proportion — not magic words. "All" does not mean add. Teach the structure. Vary the cover story so they match the type, not the noun.

Kitchen, money, and making can motivate, or give a second picture of a number they are already placing on a line. They do not replace the line, the equation, or this page. A child who can double a recipe and cannot place (3/2) on a line has performed, not learned.

Keep practice short enough that attention holds. More pages are not more learning if the last ten are sloppy. Stop while they can still think.

6. Unaided exit ticket

Three to five minutes. One or two problems. Book closed. No hints. No "For the student." No chatbot. No looking at the worked example.

[8] Doug Rohrer, Robert F. Dedrick, and Kaleena Burgess, "The Benefit of Interleaved Mathematics Practice Is Not Limited to Superficially Similar Kinds of Problems," *Psychonomic Bulletin & Review* 21 (2014): 1323–30: grade-7 classroom crossover, interleaved 72 percent versus blocked 38 percent on a two-week delayed test. Doug Rohrer, Robert F. Dedrick, Marissa K. Hartwig, and Chi-Ngai Cheung, "A Randomized Controlled Trial of Interleaved Mathematics Practice," *Journal of Educational Psychology* 112, no. 1 (2020): 40–52: 61 percent versus 38 percent, one-month delay; interleaved assignments also space same-kind items, so treat the gain as joint with spacing. After a short blocked introduction of a new skill, mix yesterday's and last month's types.

This is the check. Tonight's helped practice, with you in the chair and a tool in reach, is not this check.[9]

Say:

"Two problems. Book closed. I won't help. That's the point."

"Wrong is useful. We'll look after."

"You don't need a perfect score. I need to see what you can do alone."

Then you are quiet again. When they finish, you look. If both are right, you are done enough for today. If one is wrong, you diagnose — which wrong answer from the parent section is this? — and you note tomorrow's first move. You do not redo the whole hour. You do not punish. You find the picture that was missing.

An exit ticket that always needs you is not an exit ticket. Fade until it doesn't.

When to stop talking

After you ask, wait. If you fill the silence, you took the problem back.

During the attempt, your job is almost nothing. Presence, not narration. If you hear yourself explaining while they write, you have moved back into the model. Stop. Let them write.

Instead of "this is the hard part" or "I was never good at this," say today's sentence: "Let's look at the line." The first sentence fills their working memory with your feeling. The second sentence teaches.

Praise a clear picture, a second method, a catch of their own error — not speed.

"You noticed 2/5 was too small. That's the number sense I want."

Stop talking when the next sentence would be a rescue. Hands in your lap. Count ten. Then one hint, or one step, and the pencil goes back.

If you remember only one move from this page, remember that one.

[9] Nicholas C. Soderstrom and Robert A. Bjork, "Learning Versus Performance: An Integrative Review," *Perspectives on Psychological Science* 10, no. 2 (2015): 176–99: current performance is "often an unreliable index" of learning. The unaided exit ticket is the check this hour trusts.

The AI rules, once

Later chapters will point here in one sentence. They will not reprint this box.

> **Child first. You hold the key. AI is optional for the adult.**
>
> The student attempts the problem before any tool is in the room. You keep the answer key — on paper, in a closed tab, in your head after the warm-up. A chatbot does not sit beside them during the attempt.
>
> Use a tool, if you use one at all, for four jobs: explain today's idea *to you* before the lesson; make extra practice with the answers hidden from the child; write a hint script you can read from; diagnose work they already produced.
>
> Never paste the worksheet, ask "what's the answer," photograph the page for a key, or leave an unsupervised chatbot in their hands during the attempt. Bastani and colleagues found that an unguarded chatbot raised practice scores and then cut the unaided exam.[10]
>
> Facts are not a language-model job. Use a tool to generate a short oral list or a practice deck. The child retrieves the fact.
>
> You can teach this entire book with no AI. Many families will. The hour still has the same shape.

If a product claims it "does 100 percent of the teaching," treat that as advertising. You still need to hear a wrong answer. You still hold the key.

A Tuesday, said plainly

Here is an illustration, not a reported family. A parent has spent five minutes placing (3/4) on a line and writing the sentence "fractions are numbers with a size." The child warms up on "two ways to make 1." The parent models one

[10] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without guardrails can harm learning: Evidence from high school mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. In a field experiment, a ChatGPT-like tool raised assisted practice grades 48 percent relative to control, then cut unaided exam grades 17 percent. A hint tutor that withheld full solutions raised practice without that exam drop. Not a homeschool study; the transferable caution is the crutch. Access date for URLs in these notes: 30 August 2026.

placement out loud, then leaves the last tick-mark blank. The child places (2/3), slowly. The parent asks, "How did you know it was less than 1?" Mixed practice: two placements, one addition they already know, one comparison. Exit ticket: "Put (5/4) on the line. I won't help." Right or wrong, the hour had a shape. Tomorrow's first move is already visible.

You can run that hour. You do not need a degree. You need today's sentence, a pencil you refuse to take, and the willingness to stop talking.

Later chapters will say: run the session as in *The Math Hour*. That sentence is this page. The idea will change. The hour will not.

Chapter 1

Number, operations, and place value

Figure 2: Counting beads, a paper with 8=3+5, and generic base-ten blocks on a kitchen table.

Why this matters

This year is not a counting song you wait out until "real math" begins. The work of these years is whole numbers, the structure of tens and ones that makes those numbers make sense, and the first algebra you will actually hear at the table: the equal sign as "the same amount."

That last sentence is the one that surprises people. Algebra, in the National Council of Teachers of Mathematics' sense, is already here as properties of operations and as the meaning of the equal sign — not as letters standing for unknowns in a high-school course.[11] If the child writes 15 for $3 + 4 + 5 = 3 + \square$, they added every number on the page. If they write 12, they added everything to the left of the equal sign. Both answers treat "=" as "do something; the total comes next." Nicole McNeil and Martha Alibali documented those two errors, and the operational habits behind them.[12] Eric Knuth, Ana Stephens, McNeil, and Alibali later connected a relational view of the equal sign to later algebra.[13] The trap is already in the room. Hearing it this year is worth the struggle.

What you are teaching, as mathematics, is the arithmetic of whole numbers inside 100, then inside 1,000, with place value as the structure that makes the arithmetic make sense. Linear measurement as iterating a length unit. Shapes with defining attributes. And the first equal-shares language that will become fractions. Halves and fourths of circles and rectangles, then halves, thirds, and fourths: that is the on-ramp. The gate itself — a fraction as a number on a line — is the next chapter. You plant the shares here.[14]

[11] National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics* (Reston, VA, 2000). Algebra is one of five content standards across four grade bands, including pre-K–2, as properties of operations and as the meaning of the equal sign — not as letters standing for unknowns in the high-school sense.

[12] Nicole M. McNeil and Martha W. Alibali, "Why Won't You Change Your Mind? Knowledge of Operational Patterns Hinders Learning and Performance on Equations," *Child Development* 76 (2005): 883–899. Three operational patterns: perform all given operations on all given numbers; the "operations = answer" visual structure; "=" means "the total." Children fail $3 + 4 + 5 = 3 + \square$ by adding all (15) or adding to the equal sign (12).

[13] Eric J. Knuth, Ana C. Stephens, Nicole M. McNeil, and Martha W. Alibali, "Does Understanding the Equal Sign Matter? Evidence from Solving Equations," *Journal for Research in Mathematics Education* 37 (2006): 297–312. A relational view of "=" is connected to later algebra.

[14] Common Core State Standards Initiative, *Common Core State Standards for Mathematics*, https://www.thecorestandards.org/assets/CCSSI_Math%20Standards.pdf, accessed 30 August 2026. 1.G.3 and 2.G.3 are the only CCSSM fraction language before the Grade 3 Number and Operations — Fractions domain. Cited here as a map of the mathematics, not as a homeschool

Grade labels in this book are sequence anchors, not birthdays. A six-year-old or an eight-year-old may sit here. A child of ten who never quite got "how many," or who still sees 27 as two numerals rather than two tens and seven ones, may sit here too. In the 2016 National Center for Education Statistics survey, Algebra I was taught that year to 41 percent of homeschooled children in the 6–8 band, and arithmetic was still taught to 31 percent of the 9–12 band.[15] The National Mathematics Advisory Panel's Finding 15 is the age-lock refusal: claims that children of a given age cannot learn certain content because they are "too young" or "not ready" have consistently been shown to be wrong.[16] Move a child who has the prerequisites. Hold the work for a child who does not. Finding 8 of the same Panel, and Greg Duncan and colleagues as cited by Robert Siegler and colleagues, is why these years are not a warm-up: kindergarten-entry knowledge predicts later mathematics for years.[17] That is a prediction. It is not a license to panic a five-year-old with a fifty-item timed test.

Treat the Common Core map as a map, not a homeschool mandate. Many families do not use it. Many states have adopted, adapted, or replaced it. What the map is good for is saying what the mathematics *is*: a ten as a bundle of ten ones; 11–19 as a ten and some ones; unknowns in all positions of add-to, take-from, put-together, take-apart, and compare situations; the equal sign as a relation; fluency within 20 and, by the end of the Grade 2 map, all sums of two one-digit numbers from memory; length as a whole number of length units with no gaps or overlaps; equal shares of identical wholes that need not have the same shape.[18] The map does not

mandate.

[15] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey* (NCES 2020-001), Table 9. Algebra I taught that year to 41 percent of the homeschooled 6–8 band; arithmetic still taught to 31 percent of 9–12.

[16] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (U.S. Department of Education, 2008), Finding 15, ERIC ED500486. Claims that children of a given age cannot learn certain content because they are "too young," "not in the appropriate stage," or "not ready" have "consistently been shown to be wrong." What is developmentally appropriate is "largely contingent on prior opportunities to learn".

[17] NMAP, *Foundations for Success* (2008), Finding 8. Greg J. Duncan et al., "School Readiness and Later Achievement," *Developmental Psychology* 43 (2007): 1428–1446, as cited by Robert S. Siegler, Greg J. Duncan, Pamela E. Davis-Kean, Kathryn Duckworth, Amy Claessens, Mimi Engel, Maria Ines Susperreguy, and Meichu Chen, "Early Predictors of High School Mathematics Achievement," *Psychological Science* 23, no. 7 (2012): 691–697. Prediction, not a first-grade curriculum experiment.

[18] CCSSM Grade 1 and Grade 2 overviews and domains 1.OA, 1.NBT, 1.MD, 1.G, 2.OA, 2.NBT, 2.MD, 2.G. The standards "do not dictate curriculum or teaching methods," and the order of topics

dictate your Tuesday. This book will not dump 180 days of worksheets, and it will not sell you a franchise.

What this band unlocks is Grade 3 multiplication and division, and fractions as numbers. Equal groups and arrays are the picture those later objects sit on. Place-value fluency through 1,000 and memorized single-digit sums are the start of the Panel's "fluency with whole numbers."[19] Skip the bundles and the equal sign, and later pages arrive as symbol soup. Skip the shares, and the next chapter's unit fraction has nothing to sit on.

You do not need to be a mathematician. You do need to understand today's idea well enough to hear 15 on $3 + 4 + 5 = 3 + \square$ as a meaning of "equals," not a cute slip, and 13 written for thirty-one as a tens-and-ones reversal, not messy handwriting.

For the parent: understand it yourself

Many adults feel rusty. That is ordinary. Five minutes of this section, then the warm-up at the end, is enough for tomorrow.

Everyday picture. Eight grapes on a plate. You can see them as five and three, or as four and four, or as seven and one. The plate is still eight. Nothing "comes next." The amount is the amount, named two ways.

Precise picture. Write $8 = 5 + 3$. Write $8 = 4 + 4$. Write $5 + 3 = 8$. Write $3 + 5 = 2 + 6$. The equal sign says both sides name the same amount. It does not say "here comes the answer." McNeil, Emily Fyfe, and colleagues showed that practicing formats such as $4 = 2 + 2$ helped children understand mathematical equivalence.[20] A diet of only $3 + 4 = 7$ trains the other reading: operations on the left, a blank on the right, "=" meaning "the total."

A ten is a bundle of ten ones. Twenty-seven is two of those bundles and seven leftover ones. It is $20 + 7$. It is not a "2" sitting next to a "7" the way a house

inside a grade is not a required teaching order.

[19] NMAP, *Foundations for Success* (2008), Critical Foundations of Algebra: fluency with whole numbers; fluency with fractions; particular aspects of geometry and measurement. Table 2: proficient addition and subtraction of whole numbers by the end of Grade 3.

[20] Nicole M. McNeil, Emily R. Fyfe, Lori A. Petersen, April E. Dunwiddie, and Heather Brletic-Shipley, "Benefits of Practicing 4 = 2 + 2: Nontraditional Problem Formats Facilitate Children's Understanding of Mathematical Equivalence," *Child Development* 82 (2011): 1620–1633. McNeil, Fyfe, and Dunwiddie (2014) is the follow-up.

number sits next to a street name. When you add 27 + 15, you are combining two tens and one ten, seven ones and five ones, and composing a new ten when the ones go past nine. Karen Fuson and Diane Briars' classic teaching study put base-ten blocks next to that work.[21] The Institute of Education Sciences intervention guide later put the same idea in four steps: choose representations that match the mathematics (ten-sticks that really are ten ones, not a chip that "stands for" ten); connect the sticks to the written numeral in the same visual field; practice until the sticks are a thinking tool; fade them when the child is accurate without them, and bring them back when a new idea needs a bridge.[22]

Counting is not yet "how many." Rochel Gelman and C. R. Gallistel described five counting principles. Three are how to count: each item gets exactly one tag; the tags always come in the same order; the last tag names how many are in the set. Two are what you may count: any discrete collection; the order of tagging does not change the total.[23] Later work, including Barbara Sarnecka and Susan Carey, and Vicente Bermejo and colleagues as usually cited, is the correction you actually need at the table: producing the last word is not the same as knowing it names the set. A child who recounts when you ask "how many?" has not yet got cardinality.[24] After a count, ask "how many?" without recounting. Hide the set and ask again.

Part-whole knowledge of numbers to 10, then 20, is number-bond work. Eight is 5 + 3 and 8 is 4 + 4 and 8 is 7 + 1, all visible in the same sitting. Katherine Canobi found that conceptually sequenced practice supported both conceptual and procedural gains; randomly ordered practice supported only modest procedural gains.[25] Sequence the ideas. Write the bond as 8 = 5 + 3, not only as 5 + 3 = 8.

[21] Karen C. Fuson and Diane J. Briars, "Using a Base-Ten Blocks Learning/Teaching Approach for First- and Second-Grade Place-Value and Multidigit Addition and Subtraction," *Journal for Research in Mathematics Education* 21, no. 3 (1990): 180–206.

[22] Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*, WWC 2021006 (IES, 2021), Recommendation 3. Connect representations that match the mathematics to notation in the same visual field; fade for accuracy, not for a calendar date.

[23] Rochel Gelman and C. R. Gallistel, *The Child's Understanding of Number* (Cambridge, MA: Harvard University Press, 1978). Five counting principles: one-to-one, stable order, cardinal (how-to-count); abstraction, order-irrelevance (what-to-count).

[24] Barbara W. Sarnecka and Susan Carey (2008); Vicente Bermejo and colleagues (2004), as commonly cited on the distinction between producing the last word and cardinality. A child who recounts when asked "how many?" has not yet got cardinality.

[25] Katherine H. Canobi, "Concept-Procedure Interactions in Children's Addition and Subtrac-

That is already the equal-sign work, sitting inside the number.

Word problems in this band are five kinds, not one. Add-to: a starting quantity, a change that increases it, a result. Take-from: a starting quantity, a change that decreases it, a result. Put-together / take-apart: two parts and a total, no action required. Compare: two quantities and a difference. The unknown can sit in any of the three slots. “There were 8. Some more came. Now there are 13. How many came?” is add-to with the change unknown. “Maya has 8. Jordan has 13. How many more does Jordan have?” is compare. The cover story can hide the structure. Keyword lists fail because the same word appears in different structures. The 2021 IES intervention guide lists why *all*, *left*, and *total* fail, and what to do instead: introduce one problem type at a time, first as a complete story with all quantities, then as a problem with one unknown; use objects to see the action; vary the cover story so the child is matching structure, not nouns.[26]

Fluency is flexible, accurate, efficient, appropriate retrieval — not a race and not optional. *Adding It Up* defined it that way.[27] The Panel’s Finding 11: computational proficiency depends on automatic recall of addition and subtraction facts, fluency with standard algorithms, and understanding of the commutative, associative, and distributive properties.[28] Grade 1 on the map wants fluency within 10 and strategies within 20: counting on, making ten, decomposing a number leading to a ten, using the relationship between addition and subtraction, creating equivalent but easier sums (6 + 7 as 6 + 6 + 1).[29] Grade 2 wants fluency within 20 using mental strategies, and, by the end of that map, all sums of two one-digit numbers from memory.[30] Those are strategy years first. Then retrieve. The IES

---

tion,” *Journal of Experimental Child Psychology* 102 (2009): 131–149.

[26]Fuchs et al., *Assisting Students Struggling with Mathematics* (2021), Recommendation 5 and Table 5.2, on keyword strategies and schema-based how-to. School-intervention guide; not a homeschool experiment.

[27]National Research Council, *Adding It Up: Helping Children Learn Mathematics*, ed. Jeremy Kilpatrick, Jane Swafford, and Bradford Findell (National Academy Press, 2001), 116. Procedural fluency: “skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.” Five strands “interwoven and interdependent.”

[28]NMAP, *Foundations for Success* (2008), Finding 11.

[29]CCSSM 1.OA strategies named: counting on; making ten; decomposing a number leading to a ten; using the relationship between addition and subtraction; creating equivalent but easier or known sums. Fluency within 10 in Grade 1; strategies within 20.

[30]CCSSM 2.OA.2: fluently add and subtract within 20 using mental strategies; by the end of Grade 2, know from memory all sums of two one-digit numbers. Cited as a map, not a homeschool law.

2021 guide's Recommendation 6 is brief timed work on material the child can already get right untimed, often with the child graphing their own score — not a high-stakes three-minute fifty-item test in Grade 1.[31] Jo Boaler's 2012 commentary and 2015 working paper attack a different object: early speed tests used to rank young children.[32] Retrieve known facts. Leave the public race out of the identity of being good at math.

Wrong answers you should be able to hear

1. *15 on 3 + 4 + 5 = 3 + □.* The child added every number on the page. McNeil and Alibali's first operational pattern: perform all given operations on all given numbers. The equal sign was a green light to compute, not a relation between two sides.

2. *12 on the same item.* The child added everything to the left of the equal sign and wrote the total in the box. The visual habit "operations = answer." The box was "where the answer goes," even though a 3 was already sitting on the right.

3. *13 written for thirty-one.* Tens and ones reversed. The child heard "thirteen" as 1 then 3, or treated two-digit numerals as concatenated digits rather than 30 + 1. This is a place-value reversal, not a handwriting problem and not a reason to wait a year for two-digit numbers.

4. *The child finishes "1-2-3-4-5," you ask "how many?," and they start over at one.* Cardinality is not yet the last word as a name for the set. The recitation was a performance. Hide the set and ask again. If they must restart, the last tag has not yet become the number.

5. *27 + 15 computed as 3 and 12, or as 212, or as 32.* These are different mistakes. The first concatenates and adds digits without place. The second concatenates the sums. The third is a ones error with the tens intact, or a tens error with the ones intact — ask which. None of them is "they cannot add." Show 27 with sticks, show 15, combine, then write.

[31] Fuchs et al. 2021, Recommendation 6 (strong evidence, 27 studies): regularly include timed activities as one way to build fluency. Brief, mixed, already-taught material. Time a child only on facts they can already produce correctly when untimed.

[32] Jo Boaler, "Timed Tests and the Development of Math Anxiety," *Education Week* / Stanford GSE (2012); Jo Boaler, *Fluency Without Fear* (Youcubed working paper, 2015). Working paper / commentary, not a journal experiment. The object of criticism is high-stakes early speed tests, not brief timing on already-known facts.

A sixth you will also hear: every 8 + 5 still starts at 1, recounting both sets. Counting on from the larger has not yet arrived. Robert Siegler's strategy-choice work documents that shift from count-all to count-on to retrieval.[33] Once they can produce 8 + 5 as 13 untimed, a brief mixed retrieval is a different object.

Five-minute parent warm-up

Do this before the lesson, on a scrap of paper, no child in the room.

Minute 1. Write 8 = 3 + 5. Say out loud: "Eight is the same amount as three and five." Write 8 = 4 + 4. Write 5 + 3 = 8. Write 3 + 5 = 2 + 6. Circle the equal signs. They all mean the same thing.

Minute 2. True or false, said aloud: 7 = 7. 4 + 3 = 5 + 2. 4 + 3 = 8. 6 + 2 = 8. You are practicing the hearing, not racing.

Minute 3. Sketch two ten-sticks and seven small marks. Write 27, then 20 + 7, then "2 tens and 7 ones." Unpack one ten into ten ones with a little drawing. That unpacking is what 27 + 5 needs.

Minute 4. Look at 3 + 4 + 5 = 3 + □. Decide, without filling the box, what 15 would mean and what 12 would mean. The correct box is 9, because both sides are 12. You are not taking a test. You are installing the two errors so you can hear them tomorrow.

Minute 5. Write one sentence you will actually say: "The equal sign says both sides are the same amount." Put the pencil down. That sentence is the lesson.

If you can do those five minutes, you are ready to sit down. The child generates. You hear.

How to teach it this week

A good math hour this week has a shape. The welcome chapter's math-hour box is the full version, including the AI rules. Here is the shape scaled to this band.

Warm-up (3–5 minutes, unaided). Two facts the child already knows. One non-canonical equation, such as 6 = 4 + □. Paper. No device.

[33] Robert S. Siegler and Shrager (1984); Siegler and Jenkins (1989): strategy-choice work documenting the shift from count-all to count-on to retrieval. The qualitative shift is used here for diagnosis.

Short model (5 minutes). One new idea, one picture, one written sentence. Sticks next to the numeral, or a number bond written four ways. You talk for a few minutes. Then you stop.

Student attempt (10–15 minutes). Three to five items. The child writes. You wait. Struggle before rescue: ask, wait, hint, then model. Leave the pencil in the child's hand.

One good question. Not "what's the answer?" A question that names the idea: "What is the equal sign saying?" "Show me the tens." "You already counted. How many?"

Mixed practice (5–10 minutes). Yesterday's type next to today's. A compare story next to a number bond with the total on the left. If the book is almost entirely blocked — most are — you pull one older item onto the page.

Exit ticket (5 minutes, unaided). Two items isomorphic to what you taught. Phone in another room. You grade against a key you hold.

That shape is a practice you impose on whatever book is on the table. It is not a 180-day plan. The Institute of Education Sciences guide for young children recommends dedicated daily time, number and operations along a progression, and talk about the world mathematically.[34]

Exact wording you can say

On the equal sign:

"The equal sign says both sides name the same amount."

"What could go in the box so both sides are the same?"

"Is 4 + 3 = 5 + 2 true or false? How do you know?"

After a count:

"You already counted. How many are there?"

"I'm hiding them. How many are under my hand?"

On place value:

[34] Douglas Frye, Arthur J. Baroody, Margaret Burchinal, Sharon M. Carver, Nancy C. Jordan, and Jonathan McDowell, *Teaching Math to Young Children*, NCEE 2014-4005 (IES, 2013). Ages 3–6: number and operations plus geometry, measurement, and data along a developmental progression; progress-monitor; talk about the world mathematically; dedicate daily time.

"Show me the two tens in 27."

"A ten is a bundle of ten ones. Let's unpack one."

"Write 27 as 20 + 7."

On a stuck addition:

"Start from the larger number and count on. I'll listen."

"Can you make a ten first? 8 + 2 is 10, then the rest."

On a word problem:

"What happened in the story? What do we know? What is hiding?"

"This is a compare story. We are finding the difference, not putting two groups together."

When you are about to take over:

"Your pencil. I'll ask a question."

Then wait. Ten seconds of silence is teaching. Grabbing the pencil is the high-help path Erin Maloney, Gerardo Ramirez, Elizabeth Gunderson, Susan Levine, and Sian Beilock found, in a first- and second-grade school-homework study, mixed with parent math anxiety to predict worse child math at year's end.[35] You cannot "help less often" as a daily homeschool strategy. The useful response is preparation, then generation. The finding is not that anxious parents should not teach. Prepare so you can hear 15. Then leave the pencil.

First problem for the student

Write this on a small card or the top of the page, exactly:

8 = 3 + □

Say: "The equal sign says both sides are the same amount. What goes in the box?"

[35] Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock, "Intergenerational Effects of Parents' Math Anxiety on Children's Math Achievement and Anxiety," *Psychological Science* 26 (2015): 1480–1488. First- and second-graders: parent math anxiety × frequent homework help predicted worse child math and more child math anxiety at year's end; when help was infrequent, no such relation; math-specific. School-homework study, not a parent-as-sole-teacher experiment.

Wait. If the child writes 11, they added 8 and 3. If they write 5, they heard the relation. If they stare, show a fully worked different item — 6 = 4 + 2, said as "six is the same amount as four and two" — then a true/false (7 = 7), then return to 8 = 3 + □. That is fading: full example, then a completion, then the item. John Sweller and Graham Cooper's worked-example effect, and Alexander Renkl and Robert Atkinson's fading, are the method. The kitchen analogue is not a new first-grade trial. It is the same method applied to a box.[36]

Later the same week, the diagnostic item:

3 + 4 + 5 = 3 + □

Hear 15. Hear 12. Hear 9. After the child has attempted, you may show a worked *incorrect* example that wrote 15 and ask what that person thought "=" meant. Generation first, then the named error. Julie Booth and colleagues found that explaining incorrect examples helped conceptual understanding of two-step equations in Algebra I.[37] Scaled down, labelled as a scaling: the child names the error. You do not narrate it as a verdict on talent.

How to fade help

Day of a new move: you model one fully. The child explains a second. A third is missing only the last written step. A fourth is conventional. When the child is accurate, stop showing full examples. Bring the sticks back when composing a ten is new, or when three-digit numbers arrive. Fading is for accuracy, not for a calendar date.

When to stop talking

After the question. After the hint. After the model of a *different* item. If you hear yourself explaining the child's current problem in real time while their pencil is idle, you have started doing the work. Stop. Point to the sticks. Point to the two sides. Wait.

---

[36] John Sweller and Graham A. Cooper, "The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra," *Cognition and Instruction* 2, no. 1 (1985): 59–89. Alexander Renkl and Robert K. Atkinson, "Structuring the Transition from Example Study to Problem Solving in Cognitive Skill Acquisition," *Educational Psychologist* 38 (2003): 15–22.

[37] Julie L. Booth, Karin E. Lange, Kenneth R. Koedinger, and Kristie J. Newton, "Using Example Problems to Improve Student Learning in Algebra: Differentiating between Correct and Incorrect Examples," *Learning and Instruction* 25 (2013): 24–34. Algebra I paper; the move is scaled down here.

Properties of operations sit in the same cluster and do not require formal names. Commutative: 3 + 5 is 5 + 3, which is already 8 = 3 + 5 next to 8 = 5 + 3. Associative: (2 + 3) + 4 is 2 + (3 + 4), which is making ten. Subtraction as unknown-addend: 8 − 3 is the same question as 3 + □ = 8. A child who can only "take away" and cannot hear subtraction as a missing part is missing that relationship. Formal names are optional. The hearing is not.

True and false equations are not a game at the end of the page. 4 + 3 = 5 + 2 is a sentence about quantities. A child who says it is false because "the equal sign is in the middle and nothing is being done" has the operational pattern. A child who says it is true because both sides are 7 has the relation. Your job is to hear which.

Bethany Rittle-Johnson and Martha Alibali showed bidirectionality on mathematical equivalence: a procedure lesson improved the concept relative to control, and a concept lesson improved procedures.[38] The later review is the paper to keep in mind when a program claims "research says concepts first": the ordering has barely been tested; the one classroom experiment that held content constant favored iteration.[39] A short pass on what "=" means, then a handful of non-canonical items, then a return to the sentence, then more items including a true/false and a box on the left. Withhold neither the idea nor the written equation for weeks.

Practice that actually builds learning

Understanding and practice grow together. Wait weeks to practice a new move and the idea stays a speech. Drill a new procedure with no picture and no "why" and you get a chant.

Blocked, for a new move. The day you introduce composing a ten, give a short set that is only that: 8 + 5, 9 + 4, 7 + 6, sticks next to the numerals. The day you introduce the equal sign as a relation, a short set of non-canonical equations. The day you introduce a compare story, a few compare stories with all quantities

[38] Bethany Rittle-Johnson and Martha W. Alibali, "Conceptual and Procedural Knowledge of Mathematics: Does One Lead to the Other?," *Journal of Educational Psychology* 91, no. 1 (1999): 175–189.

[39] Bethany Rittle-Johnson, Michael Schneider, and Jon R. Star, "Not a One-Way Street: Bidirectional Relations between Procedural and Conceptual Knowledge of Mathematics," *Educational Psychology Review* 27, no. 4 (2015): 587–597. The classroom experiment that held content constant and varied only order is Rittle-Johnson and Koedinger 2009 (sixth-grade decimals), which favored iteration.

visible first, then one unknown.

Mixed, for when to use it. Later the same week, the new move reappears between unlike problems: a compare story, a number bond with the total on the left, a length iteration, an older fact. Mixing types is the practice of choosing a strategy. The clean classroom randomized trials of interleaved mathematics are still grade 7 — Doug Rohrer, Robert Dedrick, and Kaleena Burgess, then Rohrer and colleagues in 2020.[40] This band borrows the *practice* of mixing, not those grade-7 coefficients. A purchased book that is almost entirely blocked can still be interleaved by you pulling yesterday's and last month's types onto the same page.

Brief retrieval of facts already known. Two or three minutes, mixed, after the child can produce those facts correctly untimed. Cover the answers. You listen. Russell Gersten and colleagues' school-intervention figure was about ten minutes per session on fluent retrieval; that is a school-intervention figure, not a homeschool-minutes study, and this chapter will not rewrite it as a household rule.[41] Overlearning — cramming a fact sheet until tonight's score is 100 percent — has limited additional benefit compared with meeting those facts again next week. Doug Rohrer and Kelli Taylor is the comparison.[42]

One incorrect example to diagnose. After the child has attempted 3 + 4 + 5 = 3 + □, show a worked solution that wrote 15. Ask: "What did this person think the equal sign meant?" After a reversal, show 13 labelled "thirty-one" and ask which digit is the tens. The child names the error. You do not announce a talent verdict.

Kitchen, money, and making. Talk about the world mathematically. Count the stairs. Count the forks. Halve a sandwich and say "halves." The IES young-children guide wants that talk.[43] The Panel's Finding 26 is the limit: teaching with "real-world" contexts improves performance on similar real-world items; it

[40] Doug Rohrer, Robert F. Dedrick, and Kaleena Burgess, *Psychonomic Bulletin & Review* 21 (2014): 1323–1330; Doug Rohrer, Robert F. Dedrick, Marissa K. Hartwig, and Chi-Ngai Cheung, *Journal of Educational Psychology* 112, no. 1 (2020): 40–52. Grade-7 classroom trials of interleaved mathematics. This chapter uses the practice of mixing types, not those coefficients as a Grade 1 finding.

[41] Russell Gersten et al., *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools*, NCEE 2009-4060 (IES, 2009), Recommendation 6: about 10 minutes per intervention session on fluent retrieval of basic facts. School-intervention figure, not a homeschool-minutes study.

[42] Doug Rohrer and Kelli Taylor, "The Effects of Overlearning and Distributed Practice on the Retention of Mathematics Knowledge," *Applied Cognitive Psychology* 20 (2006): 1209–1224.

[43] Frye et al., *Teaching Math to Young Children* (2013).

does not, in the small set of high-quality studies the Panel admitted, improve computation, simple word problems, or equation solving.[44] Counting the forks does not replace two tens and seven ones written next to 27. Cooking does not replace equal shares of a rectangle, and it does not place 1/2 on a line. Money word problems are a useful second picture of tens and ones the child is already being taught. They are not the first picture.

Arrays and equal groups this year. Five rows of four written as 4 + 4 + 4 + 4 + 4. Odd and even as pairing. Skip-count by 5s, 10s, and 100s as jumps on a line. These are multiplication seeds, not a times-table race. The Grade 3 demand to know products from memory is the next chapter. Skip this seed and that chapter arrives as a chant.

Equal shares this year. Partition circles and rectangles into two, three, or four equal shares. Say halves, thirds, fourths, quarters. Notice that more equal shares of the same whole make smaller pieces. Notice that equal shares of identical wholes need not have the same shape — a half as a cut through the diagonal of a rectangle, a half as a cut through the midline. The IES fractions guide's first recommendation is to build on informal sharing.[45] You are not teaching 3/4 as a number on a line yet. You are planting the share the line will need.

Length as iteration. Order three objects by length. Express length as a whole number of length units, no gaps, no overlaps. A child who lays paper clips along a pencil with spaces between them will report too small a number. Overlaps, too large. The number is a count of units. The unit has to tile. That sentence is the on-ramp to area and to a fraction as a length on a line. Grade 2 makes the units standard — inches, feet, centimeters, meters — and asks the child to relate the size of the unit to the number of iterations. Number-line diagrams: a jump of 8 from 12 is 20, which is 12 + 8, which is already the equal sign drawn as a path.

A weekly shape: day of a new strategy, a short blocked set. Same week: mixed retrieval of older facts. Later this week and next: the new strategy between unlike problems. Monthly: an unannounced mixed quiz without notes. You impose that shape on the book you already have.

[44] NMAP, *Foundations for Success* (2008), Finding 26.

[45] Robert Siegler, Thomas Carpenter, Francis Fennell, David Geary, James Lewis, Yukari Okamoto, Laura Thompson, and Jonathan Wray, *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*, NCEE 2010-4039 (IES, 2010), Recommendation 1: build on informal fraction experiences of young children (sharing, fair shares). Recommendation 2 (number lines as a central tool) is Chapter 2 of this manual.

For the student

You are learning how numbers are built, and what the equal sign is saying.

The equal sign means both sides are the **same amount**. It does not mean "here comes the answer."

Eight grapes can be five and three. They can be four and four. The plate is still eight. We write that as 8 = 5 + 3 and as 8 = 4 + 4. Same amount, two names.

Tiny worked example

Look at this:

6 = 4 + □

The left side is 6. The right side has to be the same amount as 6. Four and what make 6? Two. So the box is 2.

Check: left side 6. Right side 4 + 2, which is 6. Same amount. True.

Two tries

1. 8 = 3 + □
2. 5 + 3 = 4 + □

For the second one, both sides have to be the same amount. Five and three is 8. So four and what make 8?

Write your answers. Check each one by saying "same amount" about both sides.

Explain it back

Tell someone at the table, in your own words, what the equal sign means. Then point to 8 = 3 + 5 and say why it is true. Then point to 4 + 3 = 8 and say whether it is true or false, and why.

Challenge

3 + 4 + 5 = 3 + □

Some people write 15. Some people write 12. What would those two answers mean? What number actually belongs in the box? How do you know both sides are the same amount?

You are allowed to struggle. You may use sticks or a drawing. You write. If you get stuck, ask for a hint — not the answer. Then try again.

A ten is a bundle of ten ones. Twenty-seven is two bundles and seven ones. When someone writes 13 for thirty-one, the tens and the ones have swapped places. You can catch that. Show the bundles. Then write.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The count restarts every time you ask "how many?"

Cardinality is not yet installed. Next move: smaller sets, five or fewer. Count. Ask "how many?" without recounting. Hide the set. Ask again. Count the same row left to right, then right to left, and ask whether the number changed (order-irrelevance). Mix kinds — bears and claps — so counting is not only one toy. Subitizing small sets, seeing 1, 2, or 3 without tagging, is the other route to numerosity, so counting is not the only one.[46] If this is still the bottleneck after several weeks of short daily work, stay here. Tens-and-ones work on top of a recitation will not hold. A human tutor who will sit with sets, not worksheets, is a reasonable next step if you have tried the small-set work and the restart continues.

2. Two-digit numbers reverse, or 27 + 15 is concatenated.

Place value is still two numerals side by side. Next move: proportional ten-sticks in the same visual field as the written 27. Say "two tens and seven ones." Write 20 + 7. Unpack a ten when you need it. Mentally 10 more and 10 less, which is the tens place moving while the ones stay put — a clean check of whether the bundle is there. Slow down the two-digit written work until the bundle is hearable. Go ahead to adding within 100 once 10 more / 10 less is easy and 13-for-thirty-one has stopped. A tutor is useful if reversals persist after the sticks-and-numeral work is consistent, especially if the child also still recounts sets.

3. 15 or 12 on 3 + 4 + 5 = 3 + □, after you have already taught the sentence.

The operational equal sign is entrenched. Next move: more non-canonical forms, not more pages of 3 + 4 = 7. Write 8 = 3 + 5 every day this week. True/false

[46] Subitizing small sets (1–3, then 4–5) so that counting is not the only route to numerosity.

sentences. A box on the left. Number bonds with the total on the left. After an attempt, one incorrect example to explain. Slow down the "new" arithmetic until the relation is hearable; a child who is fast at 8 + 7 and still writes 15 on the diagnostic is not ready to skip this. Go ahead once true/false is reliable and the box on the right of $3 + 5 = \square + 6$ is 2, not 8. A tutor is optional here; the formats are cheap and you can hear them. Bring someone in if the operational reading is still the default after a couple of weeks of daily non-canonical work *and* the child is becoming distressed. Distress from a well-posed new idea is intellectual effort, the kind James Hiebert and Douglas Grouws named.[47] Distress from missing facts is a fluency problem. Distress from a parent who has taken the pencil is a different problem. Separate them.

When to slow down. Cardinality missing. Concatenation still the default for two-digit numbers. Count-all on every fact. Equal-shares language never planted. Those are brakes. The Panel's Finding 15 still applies: "not ready" because of age is the brake this book will not use. "Not ready" because the prerequisite is missing is a real brake.

When to go ahead. Cardinality reliable. Tens as bundles. Equal sign as same amount on true/false and on a box on either side. Facts within 10 fluent, strategies within 20 moving toward retrieval. Arrays as equal addends. Equal shares named. Then the next chapter's multiplicative work and fractions-as-numbers have somewhere to sit. Finding 30 of the Panel licenses faster movement for mathematically gifted students with motivation; it does not license skipping the equal sign because a child is "good at math."[48]

When to get a human tutor. You have run the small-set cardinality work, or the sticks-next-to-numerals work, or the non-canonical formats, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Video curricula exist for the same reason. Keep yourself as the person who can still hear 15 and 13-for-thirty-one. Outsourcing the hearing is the thing to avoid, not asking for help.

[47] James Hiebert and Douglas A. Grouws, "The Effects of Classroom Mathematics Teaching on Students' Learning," in *Second Handbook of Research on Mathematics Teaching and Learning*, ed. Frank K. Lester (NCTM / Information Age, 2007). Two features that support skill and conceptual understanding: explicit attention to connections; students' struggle with important mathematics. "Struggle" is intellectual effort to make sense, not abandonment.

[48] NMAP, *Foundations for Success* (2008), Finding 30.

Hear a thin strand without ranking the strands. *Adding It Up* named five, interwoven: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition.[49] The Panel said ranking them is misguided.[50] A child who can recite 8 + 7 but cannot say why 8 + 7 equals 7 + 8 is missing one strand. A child who can draw the bond and cannot retrieve 8 + 7 inside 18 + 7 is missing another. Hearing which is thin is enough.

For a child already struggling with this content, explicit instruction — clear models, extensive practice, think-alouds, extensive feedback — has consistently positive effects.[51] Productive failure is a different comparison: generation-then-instruction on a new concept when priors exist.[52] If the child is missing cardinality, model counting a set, then guided practice, then independent. If the child has cardinality and the task is a first composed ten, ten to twenty minutes of generation with what they already know, then a clear model, is the second sequence, scaled and labelled as a scaling. Those two literatures are not the same sitting.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The short version: the child attempts first. You hold the answer key. A tool may explain today's idea *to you*, make extra practice with answers on a separate page you keep, write a hint script you read aloud, or help you diagnose work the child has already done. The child writes on paper.

A language model will happily invent 7 + 8. Facts are not a chatbot job. Generate an oral script if you like. Then your ears do the drill, or a printed addition chart you trust does. Hamsa Bastani and colleagues found that an unguarded ChatGPT-like tutor raised assisted practice scores and then cut the unaided exam; a hint-only

[49] NRC, *Adding It Up* (2001), 116.

[50] NMAP, *Foundations for Success* (2008), Finding 10.

[51] NMAP, *Foundations for Success* (2008), Finding 27; Fuchs et al. 2021, Recommendation 1 (systematic instruction, strong evidence, 43 studies). Finding 27 notes that this "does not mean that all of a student's mathematics instruction should be delivered in an explicit fashion."

[52] Manu Kapur, "Productive Failure in Learning Math," *Cognitive Science* 38, no. 5 (2014): 1008–1022. Problem-solving first, then instruction, produced comparable procedural knowledge to instruction-first but greater conceptual understanding and transfer. A different comparison from Finding 27.

tutor did not hurt the exam.[53] Nicholas Soderstrom and Robert Bjork had already named the confusion: performance during practice is often an unreliable index of learning.[54] "Finish your facts with a solver so we can go to soccer" is that pattern with smaller numbers.

Paste the worksheet, "what's the answer," photo-to-key, and an unsupervised chatbot during the attempt stay out of the hour. A six-year-old does not live in an open-web agent. You using your own adult account to prepare, with the child not in the loop until the page is on paper, is the lane the math-hour box describes.

Manipulatives are a tool too. Ten-sticks that are ten ones, in the same visual field as the numeral, with you talking about the bundle, then faded. A meta-analysis by Kira Carbonneau, Scott Marley, and James Selig found a modest average advantage for manipulatives versus symbols only, larger under guidance.[55] The conversation is the variable. The bin of bright blocks left on the table is not.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 1" book is a publisher's scope, not a legal grade. Singapore 1A, Saxon 1, and Math-U-See Alpha can place the same child in three different rooms on the same afternoon, because they cut the grain differently.[56] Placement tests exist because the labels lie. Choose by fit. This book

[53] Hamsa Bastani et al., "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. Unguarded tutor raised assisted practice and cut the unaided exam; hint-only tutor did not hurt the exam. Not a homeschool study. The pattern is the sitting condition.

[54] Nicholas C. Soderstrom and Robert A. Bjork, "Learning Versus Performance: An Integrative Review," *Perspectives on Psychological Science* 10, no. 2 (2015). Current performance is "often an unreliable index" of learning.

[55] Kira J. Carbonneau, Scott C. Marley, and James P. Selig, "A Meta-Analysis of the Efficacy of Teaching Mathematics with Concrete Manipulatives," *Journal of Educational Psychology* 105, no. 2 (2013): 380–400. 55 studies, kindergarten through college, manipulatives versus abstract symbols only. Overall mean d approximately 0.37. Larger on retention than on transfer; high guidance associated with higher retention and problem solving. Overall mean, not a Grade 1 effect size.

[56] Singapore Math Inc.: number bonds and mastery within a level. Saxon Homeschool: skill-level rather than grade-level names. Math-U-See: Alpha is single-digit addition and subtraction; Beta is multi-digit. Beast Academy / Art of Problem Solving: generally start about a level behind calendar grade because of problem difficulty. No homeschool-curriculum randomized trial comparing them was opened for the research behind this book. Choose by fit.

names those programs as options. It does not rank them.

Checklist before moving on

- After a count, the child can say how many without restarting, including when the set is hidden.
- A ten is a bundle. 27 is 20 + 7, shown with sticks next to the numeral, then written. 13 is not what the child writes for thirty-one.
- The equal sign means the same amount. $8 = 3 + 5$ is a normal sentence in the house. True/false equations are reliable. On $3 + 4 + 5 = 3 + \square$ the child writes 9, or can explain why 15 and 12 are the two operational misses.
- Unknowns show up in all three slots of add-to, take-from, put-together/take-apart, and compare stories, within 20, then within 100. The child matches structure, not a magic word.
- Strategies within 20 are in use: counting on, making ten, doubles plus one, subtraction as a missing addend. Facts within 10 are fluent. Single-digit sums are moving toward memory, mixed, after they are right untimed. A timer, if you use one, is brief and only on facts already known.
- Length is a count of units with no gaps or overlaps. A number-line jump matches an addition sentence.
- Shapes are told by defining attributes (a triangle is closed and three-sided), not by a prototype that rejects an inverted triangle.
- Equal shares: halves, thirds, fourths. More shares, smaller pieces. Equal shares of identical wholes need not have the same shape.
- Arrays up to 5×5 written as sums of equal addends. Odd and even as pairing.
- The child can explain a strategy in ordinary language. You can hear a wrong answer and ask a good question without taking the pencil.

If most of that list is true, go on to multiplicative reasoning and fractions as numbers, even if the birthday says otherwise. If the birthday says “third grade” and cardinality is still a restart, stay. The next chapter is the gate. This chapter’s job was to make the bundles, the equal sign, the facts, and the shares hearable.

A path through mathematics is the promise. A percentile is not.

Chapter 2

Multiplicative reasoning and fractions

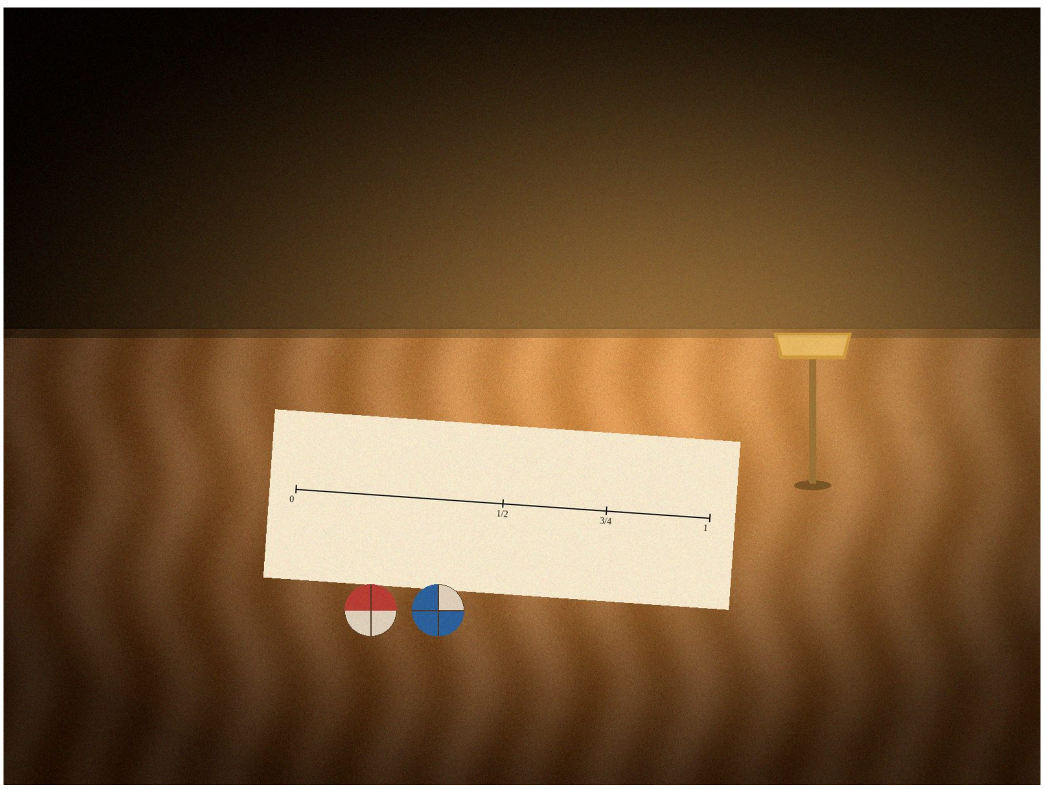

Figure 3: A paper number line marked 1/2 and 3/4, and generic fraction circles on a wooden table.

Why this matters

This is the heaviest elementary chapter in the book, and it is the gate.

The National Mathematics Advisory Panel called knowledge of fractions "the most important foundational skill not currently developed among American students."[57] The Panel had named three Critical Foundations of Algebra — fluency with whole numbers; fluency with fractions, including decimals, percents, and negative fractions; and particular aspects of geometry and measurement — and then said, of those three, which one U.S. students most often lack.[58] A 2007 survey of 743 Algebra I teachers, commissioned by the Panel, rated incoming preparation as poorest in rational numbers and operations with fractions and decimals.[59] That survey is the 2007 document the Panel used. It is not a claim about every classroom this year. It is why this chapter is worth the struggle.

You will see the second document on a page the child has just finished: 1/2 + 1/3 = 2/5. The error looks like a forgotten rule. It is a magnitude failure. The child has added the counts that are visible — two numerators, two denominators — as if the symbols were whole numbers sitting side by side. Yujing Ni and Yong-Di Zhou named that over-extension whole-number bias.[60] The IES fractions practice guide named missing magnitude as often at the root of adding numerators and denominators and of not seeing fractions as numbers.[61] Pies introduce part-whole.

[57] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (U.S. Department of Education, 2008), ERIC ED500486. "Knowledge of fractions is the most important foundational skill not currently developed among American students."

[58] NMAP, *Foundations for Success* (2008), Critical Foundations of Algebra: fluency with whole numbers; fluency with fractions, including decimals, percents, and negative fractions; particular aspects of geometry and measurement.

[59] NMAP, *Foundations for Success* (2008), nationally representative survey of Algebra I teachers, 743 teachers, 2007: poorest incoming preparation in "rational numbers and operations involving fractions and decimals," alongside word problems and study habits. 2008 data, not a 2026 classroom census.

[60] Yujing Ni and Yong-Di Zhou, "Teaching and Learning Fraction and Rational Numbers: The Origins and Implications of Whole Number Bias," *Educational Psychologist* 40, no. 1 (2005): 27–52. Whole-number bias includes unique successors, a finite number of entities in an interval, multiplication makes bigger, and adding the counts you see. 1/2 + 1/3 = 2/5 is the counts-you-see error.

[61] Robert Siegler, Thomas Carpenter, Francis Fennell, David Geary, James Lewis, Yukari Okamoto, Laura Thompson, and Jonathan Wray, *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*, NCEE 2010-4039 (IES, 2010), Recommendation 2. Missing

They do not by themselves teach that 3/4 is a number with a magnitude, a location on a line, a quantity you can compare with 2/3 without a trick. The Panel wrote that representing fractions on a number line is "one key mechanism linking conceptual and procedural knowledge."[62] The line is the teaching.

Robert Siegler, Greg Duncan, and colleagues found that elementary students' knowledge of fractions and of whole-number division uniquely predicted algebra knowledge and overall mathematics five or six years later, after other mathematical knowledge, general intellectual ability or working memory, and family income and education were already in the model.[63] A one-standard-deviation increase in age-10 fractions knowledge was associated with about 0.15 (United Kingdom) / 0.17 (United States) standard deviations more high-school algebra.[64] Concurrent correlations of high-school fraction knowledge with overall math were very large.[65] Those numbers are a *prediction*. They are not a classroom experiment proving that one fractions program causes algebra success. Grades 3–5 fractions work is still the gate.

What you are teaching, as mathematics, is four operations on whole numbers taken toward fluency; fractions as numbers, then operations on fractions; decimals as tenths and hundredths on the same line; area, then angle, then volume; the first quadrant of the coordinate plane.[66] Multiplication is 5×7 as five groups of seven, not a rhyme. A unit fraction $1/b$ is one of b equal parts; a/b is a copies of $1/b$; both are numbers on a line. Equivalence is the same point. Unlike-denominator addition is replacing with equivalent like-denominator fractions. Multiplication is

---

magnitude is "often at the root" of adding numerators and denominators and of not seeing fractions as numbers.

[62] NMAP, *Foundations for Success* (2008): representing fractions on a number line is "one key mechanism linking conceptual and procedural knowledge."

[63] Robert S. Siegler, Greg J. Duncan, Pamela E. Davis-Kean, Kathryn Duckworth, Amy Claessens, Mimi Engel, Maria Ines Susperreguy, and Meichu Chen, "Early Predictors of High School Mathematics Achievement," *Psychological Science* 23, no. 7 (2012): 691–697. British Cohort Study $N = 3{,}677$; U.S. PSID-CDS $N = 599$. Prediction finding, not a fractions-curriculum experiment.

[64] Siegler et al., "Early Predictors" (2012). A 1-SD increase in age-10 fractions knowledge uniquely associated with a 0.15-SD (UK) / 0.17-SD (US) increase in high-school algebra, after the controls in n. 63.

[65] Siegler et al., "Early Predictors" (2012). Concurrent correlations of high-school fraction knowledge with overall math: $r = .81$ (UK) and $r = .87$ (US).

[66] Common Core State Standards Initiative, *Common Core State Standards for Mathematics*, Grade 3–5 overviews, https://www.thecorestandards.org/assets/CCSSI_Math%20Standards.pdf, accessed 30 August 2026. Map, not a homeschool mandate.

scaling — the sentence that kills “multiply makes bigger.” Division of a fraction by a fraction is not a Grade 5 requirement on the map; it is Grade 6.[67] Cooking can motivate. Cooking does not replace the line.

Grade labels remain sequence anchors, not birthdays. A child may sit this band at calendar grade 2 or calendar grade 6. The 2016 NCES stretch still holds.[68] The Panel’s Finding 15 still refuses the age lock if prerequisites are present.[69] Move a child who has last chapter’s bundles, equal sign, facts, and equal shares. Hold the gate for a child who does not. Skipping fractions because a child is “good at math” is the error Finding 30 does not license.[70]

A “grade 5” book is a publisher’s scope, not a legal grade. Math-U-See Epsilon is a fractions *level*, not fifth grade. Saxon 5/4 and 6/5 are skill names. Beast Academy generally starts about a level behind because of problem difficulty. Dimensions Math 6 is a Grade 6 text with some pre-algebra topics, not a pre-algebra course.[71] Placement tests exist because the labels lie. Place by skill.

You do not need to be a mathematician. You do need to hear 2/5 as a size mistake, and 4 × 1/2 as a shrink, before any tool prints a correct least-common-denominator path you never see.

For the parent: understand it yourself

Many adults feel rusty on fractions. Five minutes at the end of this section is enough for tomorrow. Last chapter installed the equal sign as the same amount and 8 = 3 + 5. Keep both. This chapter writes 1 = 2/3 + 1/3 on the same terms.

[67] CCSSM 5.NF.7 limits division to unit-fraction ÷ whole and whole ÷ unit-fraction. Division of a fraction by a fraction is Grade 6 (6.NS).

[68] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey* (NCES 2020-001), Table 9. Algebra I taught that year to 41 percent of the homeschooled 6–8 band; arithmetic still taught to 31 percent of 9–12.

[69] NMAP, *Foundations for Success* (2008), Finding 15.

[70] NMAP, *Foundations for Success* (2008), Finding 30. Acceleration requires prerequisites; Siegler et al. 2012 is why those include fractions.

[71] Singapore Math Inc.: Dimensions Math 6 “includes some Pre-algebra topics, but is not a Pre-algebra text in the traditional sense; it is a Grade 6 text.” Saxon Homeschool split names (Math 5/4, 6/5) are skill-level. Math-U-See Epsilon is a fractions level. Beast Academy / Art of Problem Solving generally recommends starting about a level behind. No homeschool-curriculum randomized trial comparing them was opened for the research behind this book.

Everyday picture. A paper strip is one whole. Fold it in half. That fold is the number 1/2 — not a pizza slice that happens to be drawn, a location halfway from 0 to 1. Fold each half in half. The third fold from 0 is 3/4. Same strip. Same line. 3/4 is three copies of 1/4.

Precise picture. A unit fraction $1/b$ is one part when a whole is partitioned into b equal parts. a/b is a copies of $1/b$. Those two sentences are the mathematics.[72] Partition the interval from 0 to 1 into b equal parts. Each part is $1/b$. Iterate from 0. Three iterations of 1/4 land at 3/4. That point is the number 3/4. The same point is 0.75, and 6/8, and 75/100. Equivalence is the same point, not a butterfly.

The IES fractions guide — Siegler, Thomas Carpenter, Francis Fennell, David Geary, and colleagues — Recommendation 2: help students recognize that fractions are numbers that expand the number system; use number lines as a central representational tool.[73] The 2021 intervention guide, Fuchs and colleagues, Recommendation 4, is the same move for struggling elementary students, rated strong evidence.[74] Three documents, one representation.

Siegler, Clarissa Thompson, and Michael Schneider supply the cognitive claim. Numerical development is the progressive extension of the mental number line to new classes of number, and the learning that many whole-number properties are not properties of number. Fractions are the first large opportunity to learn that.[75] Whole numbers have unique successors. Between 3 and 4 there is no whole number. Between 3 and 4 there are infinitely many fractions. Multiplication of whole numbers by a whole number greater than 1 makes bigger. Multiplication by 1/2 makes smaller. Addition of "pieces" the child can see is not addition of magnitudes. A child who cannot place 3/4 on a line between 0 and 1, nearer to 1 than to 0, is the child who will write 1/2 + 1/3 = 2/5, because 2/5 is what you get when you add the counts you see, and there is no magnitude in the room to reject it.

[72] CCSSM 3.NF.1–2, limited to denominators 2, 3, 4, 6, and 8: unit fraction $1/b$; a/b as a copies of $1/b$; fractions as numbers on the number line.

[73] Siegler et al., *Developing Effective Fractions Instruction* (2010), Recommendation 2.

[74] Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*, WWC 2021006 (IES, 2021), Recommendation 4 (number lines, strong evidence, 14 studies).

[75] Robert S. Siegler, Clarissa A. Thompson, and Michael Schneider, "An Integrated Theory of Whole Number and Fractions Development," *Cognitive Psychology* 62 (2011): 273–296. Magnitude estimation of fractions is closely related to correct fraction arithmetic.

Equal shares from last chapter are the on-ramp. If they were skipped, diagnose that first. Unit fractions, then copies, then the line, then equivalence as the same point: 1/2 = 2/4 = 3/6 because they land on the same mark. Whole numbers as fractions: 3 = 3/1 = 6/2, same point as the whole number 3. Same whole matters. 3/4 of a small bar and 3/4 of a large bar are not the same magnitude.

Compare with benchmarks 0, 1/2, and 1 before any cross-multiply trick. The 2021 intervention panel advises against cross-multiply-to-compare during intervention because the trick does not build magnitude.[76] If the child can already place both fractions and estimate against 1/2, a later compression is iteration. The trick as the *first* comparison is a procedure with no picture.

Multiplication in this band is groups, arrays, area, and facts from memory. 5×7 is five groups of seven. Division is two situations: how many in each group (partitive), and how many groups (measurement). Both are unknown-factor problems. $5 \times \square = 35$. By the end of the Grade 3 map, all products of two one-digit numbers from memory — a map, not a homeschool law.[77] Area is the array made measurement: tile with unit squares, then name rows of unit squares, then the formula. The distributive property via a 5-by-7 rectangle split as 5-by-5 plus 5-by-2 is the same property later used to multiply 14×23, and later still to multiply polynomials.

"Multiply makes bigger; divide makes smaller" is not cute folklore. It is a whole-number property that fails for multipliers and divisors less than 1. Ni and Zhou reviewed that over-extension.[78] Common Core 5.NF.5 exists specifically to break it: multiplication as scaling. A fraction greater than 1 enlarges. A fraction less than 1 shrinks.[79] $4 \times 1/2$ is four copies of one-half, which is 2, which is smaller than 4. The test is not only "did they get 2." The test is "did they flinch, and can they say why."

Unlike-denominator addition, on the Grade 5 map, is replacing with equivalent like-denominator fractions. 1/2 is 3/6. 1/3 is 2/6. Three copies of 1/6 plus two

---

[76] Fuchs et al. 2021, Recommendation 4 obstacles: the intervention panel advises against cross-multiply-to-compare during intervention because the trick does not build magnitude. Example 4.5: equivalent fractions as the same point on a line.

[77] CCSSM 3.OA.7: fluently multiply and divide within 100; by the end of Grade 3, know from memory all products of two one-digit numbers. Cited as a map, not a mandate. NMAP Finding 11: automatic recall of multiplication and related division facts to free working memory.

[78] Ni and Zhou, "Whole Number Bias" (2005). Stella Vosniadou, Xenia Vamvakoussi, and Irini Skopeliti treat "multiply makes bigger" as a central conceptual-change problem.

[79] CCSSM 5.NF.5: multiplication as scaling. A fraction greater than 1 enlarges; a fraction less than 1 shrinks.

copies of 1/6 is five copies of 1/6, which is 5/6, which is the point you already estimated past 1/2. The procedure without the estimate is a chant. The estimate without a line is a slogan. Iterate.

Decimals are the same numbers in place-value clothes. 0.3 is 3/10, on the same line as 3/10. "More digits, bigger" — 0.25 > 0.3 because 25 > 3 — is whole-number bias in decimal clothes. Bethany Rittle-Johnson, Siegler, and Alibali used decimal-fraction items because representation errors of this kind mediate procedure errors.[80] Trailing zeros do not move the point.

Mixed numbers are not a separate species. 1 3/4 is 1 + 3/4, which is 4/4 + 3/4, which is 7/4, which is seven copies of 1/4, which is a point between 1 and 2. Adding mixed numbers with like denominators is Grade 4 work on the map; with unlike denominators, Grade 5. The failures are predictable. Adding the whole numbers and the fractions separately without regrouping when the fraction sum exceeds 1. Treating 1 3/4 as 13/4 by concatenating — the same habit that wrote 13 for thirty-one last chapter. The line catches it: 13/4 is 3 1/4, a different point.

Invert-and-multiply is not a Grade 5 requirement. Fraction ÷ fraction is Grade 6. You still have to be able to hear it, because many books arrive early, and because Liping Ma's comparison is the usual reminder that U.S. teachers often can execute the chant and generate few explanations.[81] 1/2 ÷ 1/4: how many one-fourths fit in one-half? Two. Invert 1/4, get 4/1, multiply, get 2. The 2 matches the measurement meaning. A child who inverts the 1/2 instead has executed a chant on the wrong factor. IES 2010 Recommendation 3: help students understand why computational procedures with fractions make sense.[82] The compression is allowed. The compression as the only meaning is not.

Multi-digit multiplication is the area model grown up. 23 × 14 is (20 + 3) × (10 + 4), four partial products, four rectangles. The standard algorithm is a compression of those partial products, not a replacement for them. The live debate is when

[80] Bethany Rittle-Johnson, Robert S. Siegler, and Martha W. Alibali, "Developing Conceptual Understanding and Procedural Skill in Mathematics: An Iterative Process," *Journal of Educational Psychology* 93 (2001): 346–362. Decimal items such as 0.25 > 0.3 because 25 > 3, or 0.25 > 0.250 because it "looks longer."

[81] Liping Ma, *Knowing and Teaching Elementary Mathematics* (Mahwah, NJ: Lawrence Erlbaum, 1999). Comparison usually cited: U.S. teachers generating few or no explanations of invert-and-multiply versus Japanese and Chinese teachers generating two or three. Discussed in Siegler et al. 2012.

[82] Siegler et al., *Developing Effective Fractions Instruction* (2010), Recommendation 3: help students understand why computational procedures with fractions make sense.

the compact algorithm arrives versus how long to stay in extended strategies. The Panel wants automaticity *and* properties. The map places the standard algorithm for multi-digit addition and subtraction as Grade 4 fluency, multi-digit multiplication as Grade 5 fluency, and the division algorithm as Grade 6 fluency. A short conceptual pass on the four rectangles, then a handful of problems, then a return to the model, then more problems. Withhold neither the picture nor the general method.

Wrong answers you should be able to hear

1. *1/2 + 1/3 = 2/5.* Adding the counts you see. 2/5 is less than 1/2. 1/2 + 1/3 cannot be less than 1/2. Name the magnitude failure, not "they forgot LCD." Grade 5.NF.2 is explicit about that estimate: reject 2/5 + 1/2 = 3/7 because 3/7 is less than 1/2.[83]

2. *4 × 1/2 = 8, or 1/2 × 4 treated as "bigger."* Multiply-makes-bigger, whole-number bias. 5.NF.5 is the flinch. Ask whether the multiplier is greater than 1 or less than 1 before anyone computes.

3. *0.25 > 0.3, or 0.25 > 0.250 because it "looks longer."* More digits, bigger. Place both on a line marked in tenths and hundredths. 0.3 is past 1/4; trailing zeros do not move the point.

4. *1/5 > 1/4 because 5 is bigger, or 1/4 > 1/8 because fourths "sound bigger."* Same-numerator comparison sabotaged by treating the denominator as a whole-number count. More equal parts of the same whole make each part smaller. On the line, 1/5 is to the left of 1/4 between 0 and 1.

5. *5 × 7 recited with no groups, no array, and no jumps.* A chant without a strand. Ask "show me the five groups." A child who can draw the array and cannot retrieve 5 × 7 inside 3/5 of 35 is missing a different strand. Hear both. Ranking them is still misguided.

A sixth, related: 1/2 = □/8 answered as 4 because "double the 2, so double the 1," without being able to say "same point." The doubling happens to work. 1/2 = □/6 will not. The equal sign is still a relation. Non-canonical forms still help: 3/4 = 3/4, 6/8 = 3/4, 3/4 = 1/2 + 1/4.

Five-minute parent warm-up

Do this on a strip of paper and a scrap, no child in the room.

[83] CCSSM 5.NF.2: benchmark estimates; reject 2/5 + 1/2 = 3/7 because 3/7 is less than 1/2.

Minute 1. Draw a line from 0 to 1. Mark 1/2. Partition each half into two. Mark 2/4 on top of 1/2. Partition each half into three. Mark 3/6 on the same point. Say: "same point, three names."

Minute 2. Mark 1/4 and 3/4. Say: "3/4 is three copies of 1/4." Mark 5/4 to the right of 1. Fractions are not only the numbers less than one.

Minute 3. Mark 1/2 and 1/3. Estimate their join: past 1/2, short of 1. Write 1/2 + 1/3 = 2/5. Cross it out. 2/5 is less than 1/2. The join cannot live there.

Minute 4. Write 4 × 1/2. Four copies of one-half is 2. Smaller than 4, because 1/2 is less than 1. Write 3/2 × 4. Three-halves of four is 6. Larger than 4, because 3/2 is greater than 1.

Minute 5. Write the sentence you will say: "A fraction is a number. Show me where it lives on the line." Put the pencil down.

If you can reject 2/5 against a benchmark, and you can say why 4 × 1/2 shrinks, you are ready to sit down.

How to teach it this week

The math-hour box in the front of this book is the full session shape. Here it is scaled to this band.

Warm-up (3–5 minutes, unaided). Two known multiplication facts. One fraction placed on a blank 0-to-1 line. One like-denominator sum. Paper. No device.

Short model (5–8 minutes). One new idea, the line or the array in the same visual field as the symbols. You talk. Then you stop.

Student attempt (10–15 minutes). Three to five items. The child writes. Ask, wait, hint, then model a *different* item. Leave the pencil in the child's hand.

One good question. "Where does that number live on the line?" "Is this bigger or smaller than 1/2?" "Show me the five groups." "Did this enlarge or shrink — and why?"

Mixed practice. Yesterday's comparison next to today's sum. Area next to perimeter. A fact family: 5 × 7, 7 × 5, 35 ÷ 5, 35 ÷ 7. If the book is blocked, you pull an older type onto the page.

Exit ticket (5 minutes, unaided). Two items isomorphic to what you taught. You hold the key.

Exact wording you can say

On fractions as numbers:

"A fraction is a number. Show me where it lives on the line."

"1/4 is one of four equal parts. 3/4 is three copies of that 1/4."

"Same point, different names. That is what equivalent means."

On a wrong 2/5:

"Is 2/5 less than 1/2 or more? Point. Can 1/2 plus something positive land to the left of 1/2?"

"You added the numbers you could see. Those counts are not the sizes."

On scaling:

"Is the multiplier greater than 1 or less than 1? What should happen to 4?"

"Four copies of one-half. Draw them."

On multiplication:

"Five groups of seven. Show me the groups. Now the array. Now write 5 × 7 = 35."

"How many in each group, or how many groups? Those are two different questions."

On decimals:

"0.3 is three tenths. Mark it on the same line as 3/10."

"Extra zeros at the end do not move the point."

On the equal sign, still:

"Both sides the same amount. 1/2 = □/8 is a same-point question, not a 'do something.' "

Maloney and colleagues' high-help path still applies.[84] Prepare. Then the child generates. The constructive response is not "anxious parents should not teach." It is this week's idea, heard well enough that you can wait.

Word problems on operations are still schemas. "Altogether" does not mean add. "Of" does not always mean multiply. "Shared" does not always mean divide. Estimate first against a benchmark. Then represent. Then compute. Then check the estimate. A recipe that doubles 3/4 cup is a second representation of 2 × 3/4. It is not the line, and it is not the schema.

First problem for the student

Draw a line. Mark 0 at the left and 1 at the right. Write:

Put 3/4 on this line.

Say: "3/4 is three copies of 1/4. Partition the whole into four equal parts. Iterate three times from 0. That point is 3/4."

Wait. If the child marks something near 3 on a 0-to-4 count of tick marks without equal parts, they are counting hash marks, not iterating equal lengths. If they mark halfway, they have 1/2. If they mark just left of 1, they are close; ask them to partition first, then count copies.

A fully worked different item, if they cannot start: 2/4 on a fresh line, said as "two copies of 1/4, which is the same point as 1/2." Then return to 3/4.

Later the same week, the diagnostic:

1/2 + 1/3 = □

Hear 2/5. After the attempt, show a worked incorrect solution that wrote 2/5 and ask what that person thought the symbols were. Then mark 1/2 and 1/3, estimate past 1/2, replace with sixths, add the copies.

How to fade help

New unlike-denominator addition is high-element-interactivity material. Sweller and Cooper's worked-example effect is largest there: a fully worked solution first, then a second the child must explain, then a completion missing the last step, then

[84] Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock, "Intergenerational Effects of Parents' Math Anxiety on Children's Math Achievement and Anxiety," *Psychological Science* 26 (2015): 1480–1488. School-homework study, first- and second-graders; the high-help path still names the sitting temptation in this band.

a conventional problem.[85] Include the incorrect 2/5. Once the child can execute the replacement-and-add, stop showing full examples. Expertise reversal is real. Watching you do twenty problems in real time is not studying a worked example. The child has to study it, then generate.

Connect concrete to notation in the same visual field, then fade. Length models for fractions: a paper strip folded into four, laid on a drawn 0-to-1 line, 3/4 written at the third fold. Unit squares for area. Unit cubes for volume. Fuchs Recommendation 3 is that connection, not a bin of tiles left on the table.[86] Fade when the child can mark 3/4 on a blank line without the strip. Bring the strip back when mixed numbers or unlike denominators need a bridge.

Word problems are still schemas, not keywords. Equal groups, arrays, measurement quantities, then, in Grade 4, multiplicative comparison: "times as many," distinguished from "3 more than." Asha Jitendra and colleagues' third-grade trial of schema-based instruction moved trained word-problem items; the district test moved less; retention at eight weeks was not significant.[87] The sitting how-to that survives: one type at a time; complete story first; then one unknown; vary the cover story; match structure, not a magic word.

When to stop talking

After "where does it live?" After "bigger or smaller than 1/2?" If you hear yourself narrating the least-common-denominator path while the child's pencil is idle, stop. Point to the line. Wait.

Iterate concepts and procedures in the same week. Rittle-Johnson, Michael Schneider, and Jon Star: it is a myth that it is a one-way street from conceptual knowledge to procedural knowledge.[88] Rittle-Johnson and Kenneth Koedinger's classroom

---

[85] John Sweller and Graham A. Cooper, "The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra," *Cognition and Instruction* 2, no. 1 (1985): 59–89. Alexander Renkl and Robert K. Atkinson, *Educational Psychologist* 38 (2003): 15–22, on fading.

[86] Fuchs et al. 2021, Recommendation 3 (strong evidence, 28 studies). Length models for fractions; unit squares for area; unit cubes for volume; same visual field; fade for accuracy.

[87] Asha K. Jitendra et al., "A Randomized Controlled Trial of the Impact of Schema-Based Instruction on Mathematical Outcomes for Third-Grade Students with Mathematics Difficulties," *The Elementary School Journal* 114, no. 2 (2013): 252–276. Word-problem posttest $g = 0.46$; district achievement $g = 0.34$; no significant retention at 8 weeks.

[88] Bethany Rittle-Johnson, Michael Schneider, and Jon R. Star, "Not a One-Way Street: Bidirectional Relations between Procedural and Conceptual Knowledge of Mathematics," *Educational Psychology Review* 27, no. 4 (2015): 587–597.

experiment that held content constant was sixth-grade decimals and favored cycling.[89] A short pass on what the symbols mean, a handful of problems, a return to the idea, more problems. Withhold neither.

Practice that actually builds learning

Blocked, for a new move. The day you introduce unit fractions on a line, a short set that is only placing 1/2, 1/4, 3/4, 1. The day you introduce like-denominator addition, a short set of copies of the same unit fraction. The day you introduce a 5-by-7 array as covering, only covering.

Mixed, for when to use it. Later the same week: like-denominator addition next to a comparison against 1/2. Multiplication next to division as soon as both exist. Area next to perimeter — same numbers, different quantity. "Plot this pair" next to "name the pair of this point." Mixing is the practice of choosing. Rohrer's grade-7 coefficients stay in the next chapter. The practice of mixing transfers, labelled as transfer.[90]

Brief retrieval of facts already known. Products of one-digit numbers, mixed with the related quotients, after they are right untimed. Sometimes against a brief clock. The child may graph their own score. A public ranked Mad Minute is Boaler's object, and it is not what the IES recommendation describes.[91] Frustration from missing facts is a fluency problem, not productive struggle. A language model will happily hallucinate 7 × 8. Generate a script. Your ears do the drill.

One incorrect example to diagnose. 1/2 + 1/3 = 2/5, already named. Also 4 × 1/2 = 8. Also 0.25 > 0.3. Also a 3-by-4 rectangle labelled "12" with no unit squares. The child names what the person thought. You do not announce a talent verdict.

[89] Bethany Rittle-Johnson and Kenneth R. Koedinger, "Iterating between Lessons on Concepts and Procedures Can Improve Mathematics Knowledge," *British Journal of Educational Psychology* 79 (2009): 483–500. Sixth-grade decimals; iteration beat a concepts-first block on procedures that transfer.

[90] Doug Rohrer, Robert F. Dedrick, and Kaleena Burgess, *Psychonomic Bulletin & Review* 21 (2014): 1323–1330; Rohrer, Dedrick, Hartwig, and Cheung, *Journal of Educational Psychology* 112, no. 1 (2020): 40–52. Grade-7 classroom trials. This chapter uses the practice of mixing types, not those *d* values as a grade-4 finding.

[91] Fuchs et al. 2021, Recommendation 6. Jo Boaler, "Timed Tests and the Development of Math Anxiety," *Education Week* / Stanford GSE (2012); *Fluency Without Fear* (Youcubed working paper, 2015). Different objects: brief timing on already-known facts versus high-stakes early speed tests.

Kitchen, money, and making. A recipe that doubles 3/4 cup is a second representation of 2 × 3/4. Counting coins can motivate hundredths. Building with cubes can preview volume. Finding 26 still holds: real-world contexts help similar real-world items; they do not, in the Panel's small high-quality set, raise computation or equation solving.[92] A child who can double a recipe and cannot place 3/2 on a line has performed, not learned. Soderstrom and Bjork: current performance is often an unreliable index of learning.[93] Cooking does not replace the line. Money does not replace tenths on the same line as 1/10. Transfer from cooking and money to formal items, measured rather than assumed, remains an open question this book will not fill with analogy.

Tile, then array, then formula. Area of a rectangle is rows of unit squares. Volume of a right rectangular prism is layers of unit cubes. Perimeter is the distance around, a different quantity. Same perimeter, different area, and the reverse, force the choice. Angle measure is a fraction of a turn: a right angle is 90/360, one-fourth of a turn.

Decimals on the same line. 0.75, 3/4, and 75/100 are the same point. Operations to hundredths with explanation. 0.2 × 0.3 is 6/100, a shrink, because both factors are less than 1 — 5.NF.5 in decimal clothes. 0.25 + 0.3 = 0.28 is added-the-visible-tails. Hear it.

The first quadrant. Origin, ordered pairs, graph. (3, 5) is not "3 and 5" in either order. The first coordinate is distance from the vertical axis. Interleave "plot this pair" with "name this point."

A weekly shape: day of new skill, short blocked set. Same week, mixed retrieval of older facts. Later this week and next, the new skill between unlike problems. Monthly, an unannounced mixed quiz without notes.

For the student

A fraction is a **number**. It lives on a number line, the same way 3 and 8 do.

1/4 means: cut a whole into four equal parts; take one of them. 3/4 means: three copies of that 1/4. Three jumps of 1/4 from 0 land at 3/4. That point is the number.

[92] NMAP, *Foundations for Success* (2008), Finding 26.

[93] Nicholas C. Soderstrom and Robert A. Bjork, "Learning Versus Performance: An Integrative Review," *Perspectives on Psychological Science* 10, no. 2 (2015).

Tiny worked example

Put 1/2 on a line from 0 to 1.

Partition the line into two equal parts. The mark in the middle is 1/2. It is also 2/4, if you cut each half in half again. Same point. Two names.

Two tries

1. Draw a line from 0 to 1. Put 3/4 on it. Then put 1/4 on it.
2. 1/2 + 1/4 = □

For the second one, 1/2 is two copies of 1/4. Two copies plus one more copy is three copies of 1/4, which is 3/4. Check on the line: start at 1/2, jump one-fourth, land at 3/4.

Explain it back

Tell someone at the table what 3/4 means, using the words "copies" and "equal parts." Point to it on a line. Then say why 1/2 + 1/3 cannot be 2/5. (Hint: is 2/5 smaller or larger than 1/2?)

Challenge

4 × 1/2 = □

Some people write 8, because "multiply makes bigger." What goes in the box? Is the answer larger than 4 or smaller than 4? Why?

You are allowed to struggle. Draw the line. Draw four halves. Write. If you get stuck, ask for a hint — not the answer.

The equal sign still means the same amount. 1/2 = □/8 asks for a name of the same point, not for a total.

Cooking is a fine way to meet a fraction twice. It is not a substitute for the line you just drew.

If it isn't clicking

1. The child can split a pizza in words and cannot mark 3/4 on a blank 0-to-1 line.

Part-whole without magnitude. Next move: length models. Fold a strip. Lay it on a drawn line. Write the fraction at the fold. Benchmarks 0, 1/2, 1 every day this week, nothing else fancy. Slow down unlike-denominator addition until the line is hearable. Go ahead to equivalence as the same point once 1/2, 1/4, 3/4, and 1 are reliable. A tutor who will sit with a line, not a worksheet of pies, is useful if the mark still wanders after a couple of weeks of daily placing.

2. Facts are slow or missing, and fraction work collapses inside 3/5 of 35.

Fluency is the bottleneck, not the fraction idea. Next move: brief mixed retrieval of already-known products, fact families, the related quotients. Time only what is already right untimed. Slow the multi-digit and fraction-of-a-set work until 5×7 is available without skip-counting from 1. Go ahead on unit fractions on a line *in parallel* — that work does not have to wait for every fact. A tutor or a dumb spaced-repetition deck loaded from a known-correct table is reasonable if retrieval is still the brake after daily three-minute mixed practice.

3. 2/5 on 1/2 + 1/3, or 8 on 4 × 1/2, after you have taught the line.

Whole-number bias is still running the pencil. Next move: estimate against a benchmark *before* any procedure. Incorrect examples to explain. Scaling questions every sitting: "enlarge or shrink?" For 2/5, put 1/2 and 2/5 on the same line and ask whether adding something positive can land left of 1/2. Slow down the LCD chant until the flinch is there. Go ahead to unlike-denominator work once the child can reject 2/5 for a reason of size. A tutor is optional if you can hear the error; bring someone in if the hour has become a fight and the magnitude still is not in the room.

When to slow down. Equal-shares language never planted (go back to last chapter). Cannot place 1/2 on a line. Multiplication still only a chant with no groups. Standard algorithms with no place-value picture, or only strategies with no path to a general method. Unlike-denominator addition never reached by the end of this window, on the map the Panel used as a pace.[94]

When to go ahead. Unit fractions and copies on a line. Equivalence as the same point. Benchmarks before tricks. Like-denominator addition fluent; unlike-denominator addition in use with an estimate. Scaling: a multiplier less than 1

[94] NMAP Table 2: add and subtract fractions and decimals by the end of Grade 5; multiply and divide fractions and decimals by the end of Grade 6. Pace from curriculum comparison, not an experimental placement.

shrinks. Products from memory, mixed with quotients. Area as covering; volume as layers; first-quadrant pairs. Then ratio, signed numbers, and expressions have somewhere to sit.

When to get a human tutor. Daily line work and daily fact retrieval have been honest, the same diagnosis is still in the room, and the hour is a fight. A tutor is a release valve. Keep yourself as the person who can still hear 2/5.

For a child already behind on this topic, explicit, systematic instruction — models, practice, think-alouds, feedback — has consistently positive effects.[95] Productive failure is generation-then-instruction on a new idea when priors exist.[96] A child who cannot yet place 1/2 needs a model first. A child who has equivalence as the same point and is meeting unlike-denominator addition can spend ten to twenty minutes generating, then receive the instruction. Those two sittings are not the same.

Mathematical language is a cheap move, even for monolingual students: "greater than," not only "bigger"; "unit fraction"; "equivalent" when you mean same point; "regroup," not "borrow."[97]

Tools, including AI

Optional helpers for you. The full rules live in the math-hour box. This section stays short.

The child attempts first. You hold the key. A tool may explain unlike-denominator addition, scaling, or tenths-as-fractions *to you*; make extra practice with answers hidden on a parent page; write a hint script that names "adding the counts you see" without printing 5/6 where the child might see it; or diagnose a page the child has already produced.

The tiny problem you should be able to solve without a model before you sit down is why 1/2 + 1/3 is not 2/5. Magnitude: 2/5 is less than 1/2; the join of 1/2 and 1/3 is past 1/2.

[95] NMAP Finding 27; Fuchs et al. 2021, Recommendation 1 (systematic instruction, strong evidence, 43 studies).

[96] Manu Kapur, "Productive Failure in Learning Math," *Cognitive Science* 38, no. 5 (2014): 1008–1022.

[97] Fuchs et al. 2021, Recommendation 2 (mathematical language, strong evidence, 16 studies).

Facts are not a chatbot job. Bastani's unguarded tutor raised practice scores and cut the unaided exam; it also made arithmetic errors even when the method was right.[98] Generate an oral script for the ×7 and ×8 families. Check a sample against a printed table. Your ears do the drill.

Paste the worksheet, photo-to-key, "what's the answer," and an unsupervised chatbot during the attempt stay out. A solver will print a correct LCD path. You will never see 2/5. That hide is the failure this band cannot afford. The math-hour box is the rest.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 5" book is a publisher's scope. Epsilon is a fractions level. Saxon split names are skill names. A child of ten may sit this chapter's mathematics at calendar grade 2 or 6.

Checklist before moving on

- 5 × 7 is five groups of seven, an array, jumps on a line, and a retrieved fact. Related quotients come along. Products of one-digit numbers are known from memory, mixed, after they were right untimed.
- A unit fraction is one of *b* equal parts. *a*/*b* is *a* copies. Both live on a number line. 3/4 is nearer to 1 than to 0.
- Equivalent fractions are the same point. 1/2 = 2/4 = 3/6, pointed at, then written. Whole numbers as fractions: 3 = 3/1 = 6/2.
- Benchmarks 0, 1/2, 1 before comparison tricks. Same whole.
- Like-denominator addition is copies of a unit fraction. Unlike-denominator addition is replacement that does not move the points, then copies, with an estimate first. 1/2 + 1/3 is not 2/5, and the child can say why in size language.
- Multiplication as scaling: less than 1 shrinks; greater than 1 enlarges. 4 × 1/2 is 2, and the flinch is there.
- Division in this band: unit-fraction ÷ whole and whole ÷ unit-fraction, with measurement meaning. Fraction ÷ fraction can wait for the next chapter, on the map.

[98] Hamsa Bastani et al., "Generative AI without Guardrails Can Harm Learning," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. Unguarded tutor: practice up, unaided exam down. Arithmetic errors even when the method was right. Not a homeschool study. The pattern is the sitting condition.

- Decimals to hundredths (and thousandths on the Grade 5 map) on the same line as tenths and hundredths. 0.25 is not greater than 0.3.
- Area as covering; perimeter as distance around; volume as layers of unit cubes; angle as a fraction of a turn; first-quadrant pairs as locations.
- The equal sign still means the same amount, including inside 1/2 = □/8.
- Cooking, money, and making were second pictures. The line, the written equation, and the unaided problem did the work.
- You can hear 2/5, 8-for-4-times-a-half, and more-digits-bigger, and ask a good question without taking the pencil.

If most of that list is true, go on to ratio, signed numbers, expressions, and early functions, even if the birthday says otherwise. If the birthday says "sixth grade" and 3/4 still has no home on a line, stay. Diagnose fractions, the equal sign, and — when the next chapter arrives — signed numbers before buying Algebra I.

A path through mathematics is the promise. A percentile is not.

Chapter 3

Ratio, signed numbers, expressions, and early functions

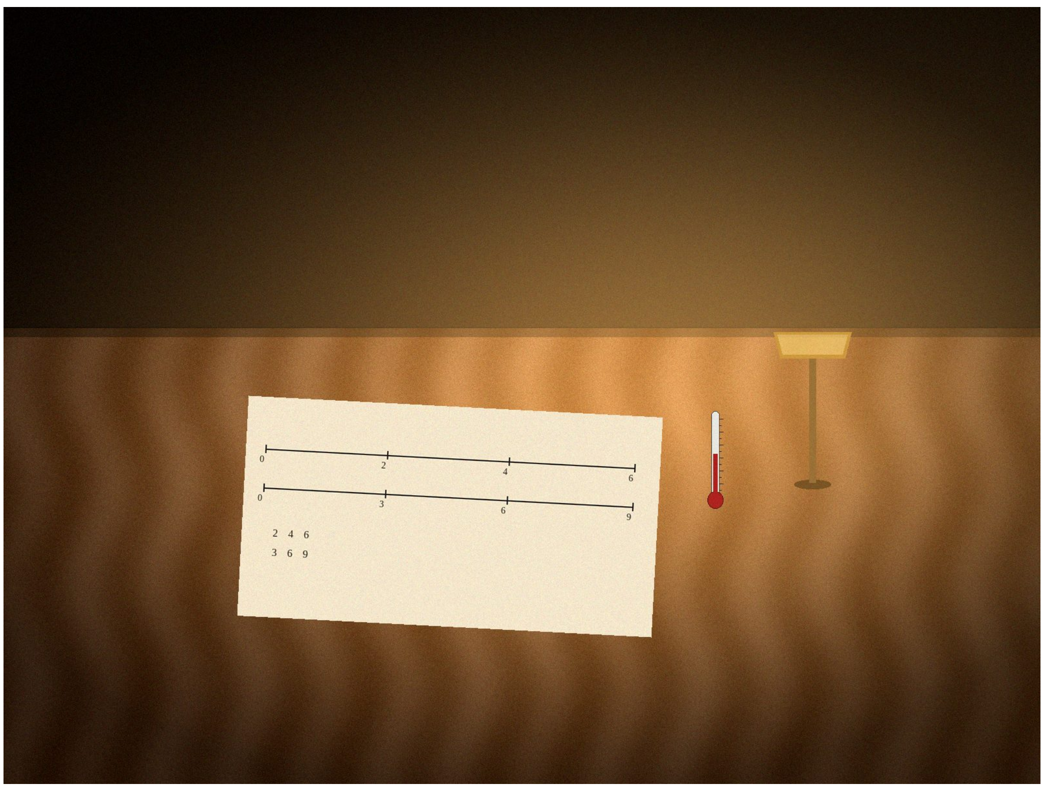

Figure 4: A double number line, a thermometer, and a ratio table drawn on paper.

Why this matters

A "grade 6" book is a publisher's scope, not a legal grade and not a finished pre-algebra. Dimensions Math 6 includes some pre-algebra topics; the publisher says it is a Grade 6 text, not a pre-algebra text in the traditional sense.[99] Beast Academy already puts integers, expressions, ratios, and percents in Level 5, and generally recommends starting about a level behind calendar grade because of problem difficulty.[100] Saxon's split names — 5/4, 6/5, 7/6, 8/7 — do not match birthdays. Math-U-See's Pre-Algebra level is when negatives and solving for the unknown are the focus: a *topic*, not a birthday.[101] A child of eleven may sit in all four of those rooms on the same afternoon. That is not a contradiction. It is four different grain-sizes of "grade."

In the 2016 NCES tables, Algebra I was taught that year to 41 percent of the home-schooled 6–8 band; arithmetic was still taught to 31 percent of the 9–12 band.[102] Grade labels remain sequence anchors. The Panel's Finding 15 still refuses the age lock if prerequisites are present.[103] Move an eleven-year-old who has fractions as numbers. Hold the work for a child who does not. Skipping signed-fraction operations because an Algebra I book is already on the shelf is the error Finding 30 does not license.[104] Last chapter was the fractions gate. This chapter is the last chance the Panel named to close the Critical Foundations before authentic algebra.[105] Di-

[99] Singapore Math Inc., Dimensions Math 6 parent-facing description: includes some pre-algebra topics but "is not a Pre-algebra text in the traditional sense; it is a Grade 6 text." Accessed in the research behind this book, 30 August 2026.

[100] Art of Problem Solving, Beast Academy FAQ: Level 5 includes integers, expressions, ratios, rates, percents, square roots, and exponents; "we generally recommend that students start about a level 'behind' their current grade level" because of problem difficulty.

[101] Houghton Mifflin Harcourt, Saxon Homeschool sequence: Math 5/4, 6/5, 7/6, 8/7 as skill-level names. Math-U-See / Demme Learning: Pre-Algebra is the level focused on negatives, order of operations, and solving for the unknown. No homeschool-curriculum randomized trial comparing these programs was opened for the research behind this book. Choose by fit.

[102] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey* (NCES 2020-001), Table 9.

[103] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (U.S. Department of Education, 2008), Finding 15, ERIC ED500486.

[104] NMAP, *Foundations for Success* (2008), Finding 30.

[105] NMAP, *Foundations for Success* (2008), Critical Foundations of Algebra, including negative fractions inside fluency with fractions. Table 2: proficient operations with positive and negative integers by the end of Grade 6; proficient operations with positive and negative fractions, and problems involving percent, ratio, rate, and proportionality, by the end of Grade 7. Pace from

agnose first. Then this band.

What the band actually is, as mathematics, is the move from arithmetic to algebra. Ratios and proportional relationships. The rational-number system, including negatives, on the same number line the child has already used for wholes, fractions, and decimals. Expressions and equations. In what the map calls Grade 8, the function concept. Informal bivariate data. Congruence, similarity, and the Pythagorean theorem.[106] Proportional reasoning is algebra readiness. Signed fractions are still inside the Critical Foundation. Grade 8 opens Functions — 8.F is the first Functions domain in K–8 — and calculus's $f'(x)$ later has nothing to attach to if that concept never forms.[107] This chapter plants both. It does not write Algebra I, and it does not write calculus.

A ratio is a relationship, not a pair of counts. 2:3 is not "two and three." Signed-number rules are compressions of moves on a line, or they are slogans. An expression is a number that depends on x, not an instruction to "do it." A function assigns to each input exactly one output; it is not "$y = mx + b$" only. Cross-multiply is a consequence, not a belief. Shopping can motivate a percent. Shopping does not replace the table, the double number line, or the graph through the origin. Cooking does not replace the line.

A compacted path through Grades 7–8 exists so that calculus is reachable in senior year. CCSSM Appendix A is explicit: compacted versions complete Grade 7, Grade 8, and the first high-school course in Grades 7–8 so a student can reach Calculus or another college-level course by senior year. "The pathways and courses are models, not mandates."[108] The Panel recommended that all *prepared* students have access to an authentic algebra course by Grade 8.[109] Compact only if the gates are open: fractions including signed fractions, integer operations, the equal

curriculum comparison, not an experimental placement.

[106] Common Core State Standards Initiative, *Common Core State Standards for Mathematics*, Grade 6–8 overviews, https://www.thecorestandards.org/assets/CCSSI_Math%20Standards.pdf, accessed 30 August 2026. Map, not a homeschool mandate.

[107] CCSSM 8.F: a function assigns to each input exactly one output. First Functions domain in K–8.

[108] Common Core State Standards Initiative / Achieve, *Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards*. Compacted traditional and integrated pathways. "The pathways and courses are models, not mandates."

[109] NMAP, *Foundations for Success* (2008): all prepared students should have access to an authentic algebra course by Grade 8. "Authentic" means the Major Topics of School Algebra, not a course that is algebra in name only.

sign as a relation. Siegler and colleagues' prediction still makes the fractions gate load-bearing.[110] Skip the gate because calculus is the destination, and the destination arrives as symbol soup.

You do not need to be a mathematician. You do need to hear 2:3 treated as a pair of counts, and "a negative times a negative is a positive" recited with no move on a line, before any tool prints a solved proportion you never see.

For the parent: understand it yourself

Last chapter's line still holds 3/4 and 0.75. This year it also holds −3/4. Keep 8 = 3 + 5 and 1 = 2/3 + 1/3. Keep the equal sign as the same amount. The letter will sit inside that relation.

Everyday picture. Two cups of juice to three cups of water. Double the recipe: four to six. Triple it: six to nine. The relationship stayed put. The counts changed. That is a ratio. A table, a double number line, a graph through the origin, and $y = (2/3)x$ are four names for the same relationship.

Precise picture. A unit rate is the ratio with a denominator of one. Twelve miles in three hours: four miles per one hour. That 4 is the k in $y = kx$ when x is hours and y is miles. It is the slope of the line through the origin. It is the number that repeats in the table: 1 hour, 4 miles; 2 hours, 8 miles; ½ hour, 2 miles. Percent is a rate per 100, not a fourth operation. 3:4 is 75 per 100 when the whole is 100.

The IES fractions guide's Recommendation 4: develop understanding of ratios and rates *before* formal proportions.[111] Cross-multiply — $a/b = c/d$ implies $ad = bc$ when b and d are not zero — is true because both ratios equal the same k, or because multiplying both sides by bd is a legal move on an equation. A child who can execute the cross and cannot say either sentence has a trick. Teach unit rate, and scaling, and the proportion equation, and ask which one fits. Flexibility is a

[110] Robert S. Siegler, Greg J. Duncan, Pamela E. Davis-Kean, Kathryn Duckworth, Amy Claessens, Mimi Engel, Maria Ines Susperreguy, and Meichu Chen, "Early Predictors of High School Mathematics Achievement," *Psychological Science* 23, no. 7 (2012): 691–697. Prediction, not a curriculum experiment.

[111] Robert Siegler, Thomas Carpenter, Francis Fennell, David Geary, James Lewis, Yukari Okamoto, Laura Thompson, and Jonathan Wray, *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*, NCEE 2010-4039 (IES, 2010), Recommendation 4: develop understanding of ratios and rates before formal proportions.

third outcome, not a luxury.[112]

Fraction divided by fraction lives here. The map withheld it from Grade 5. $1/2 \div 1/4$ is how many one-fourths are in one-half, which is 2. Invert-and-multiply is the compression of that measurement meaning. Iterate: measurement story and line first, then the rule, then practice.

Integers and signed rationals are numbers. A number and its opposite are the same distance from zero on opposite sides. Absolute value is that distance. -3 is less than -1. Place -3, -1, 0, 1/2, and 2 on the same line. If the child cannot, the rules have nothing to attach to. Grade 5's first quadrant becomes four quadrants: $(-2, 3)$ is two left and three up, a reflection of $(2, 3)$ across the y-axis.

Operations as moves, then rules as compressions. Addition of a positive is a move to the right; addition of a negative is a move to the left. $p + q$ is a distance $|q|$ from p. Subtraction is adding the opposite: $p - q = p + (-q)$. $3 - (-2)$ is a move of 2 to the right from 3, landing on 5. A child who recites "two minuses make a plus" and cannot show that move has the slogan.

Multiplication needs two justifications you can give, neither of them a scene.

The pattern: $3 \times 2 = 6$, $2 \times 2 = 4$, $1 \times 2 = 2$, $0 \times 2 = 0$, $(-1) \times 2 = -2$. Products decrease by 2 each time the first factor decreases by 1. Continue. Hold the first factor at -3 and decrease the second: $(-3) \times 2 = -6$, $(-3) \times 1 = -3$, $(-3) \times 0 = 0$, $(-3) \times (-1) = 3$. A negative times a negative is a positive because the pattern that already held for whole numbers continues. That is generalized arithmetic, which is what algebra is.

The directed-magnitude justification: multiplication by a positive stretches or shrinks and keeps direction; multiplication by a negative stretches or shrinks and *reverses* direction. Two reversals return the original direction. This is last chapter's scaling sentence, now with a sign. "Multiply makes bigger" already failed for multipliers less than 1. It fails again for negatives. Ni and Zhou's whole-number bias still operates.[113] Organize teaching so prior whole-number

[112] Michael Schneider, Bethany Rittle-Johnson, and Jon R. Star, "Relations among Conceptual Knowledge, Procedural Knowledge, and Procedural Flexibility in Two Samples Differing in Prior Knowledge," *Developmental Psychology* 47, no. 6 (2011): 1525–1538. Conceptual knowledge, procedural knowledge, and procedural flexibility each predicted gains in the others.

[113] Yujing Ni and Yong-Di Zhou, "Teaching and Learning Fraction and Rational Numbers: The Origins and Implications of Whole Number Bias," *Educational Psychologist* 40, no. 1 (2005): 27–52.

knowledge is used rather than only fought. The pattern uses it. The line is where the bias loses.

Division follows as unknown-factor. $(-6) \div (-2) = 3$ because $(-2) \times 3 = -6$. Parentheses around a negative, early and often, later save the minus-as-subtract error Julie Booth and colleagues listed among Algebra I misconceptions, citing Vlassis.[114] Hear $3 + (-2)$ as a sum of a number and a signed number, not as "three plus minus two, so subtract, I guess."

An expression is a number that depends on x. $3x + 2$, when x is 4, is 14. When x is -1, it is -1. Many students treat a variable as a single mystery number, not as a symbol that can take many values.[115] $3x + 2$ treated as "do it" is the operational equal sign, still alive, wearing a letter. McNeil and Alibali's patterns do not retire.[116] Knuth and colleagues connected a relational view of "=" to later algebra. Solving is answering which values make the two sides name the same quantity. Write $3x + 2 = 14$, try $x = 4$, both sides 14, true. Try $x = 3$, $11 = 14$, false. Then the inverse steps are a compression of that search, not a dance.

A function assigns to each input exactly one output.[117] That is the concept. Linear versus nonlinear. Table, graph, equation, verbal, in one sitting. Interpret $y = mx + b$ as a linear function, m as rate of change, b as initial value. A vertical-line test, later, is a compression of "each input exactly one output." If the table says $0 \to 1$, $1 \to 3$, $2 \to 5$, the rule is "double and add 1," the graph is a line with slope 2 and intercept 1, and the equation is $y = 2x + 1$. Then one that is not linear: $0 \to 0$, $1 \to 1$, $2 \to 4$, $3 \to 9$. The rule is "square." The graph is not a line. The function concept survived. $y = mx + b$ did not swallow it.

Slope as unit rate is 7.RP's k, now free to have a b. Similar triangles explain why

[114] Julie L. Booth, Karin E. Lange, Kenneth R. Koedinger, and Kristie J. Newton, "Using Example Problems to Improve Student Learning in Algebra: Differentiating between Correct and Incorrect Examples," *Learning and Instruction* 25 (2013): 24–34. Misconceptions include the equal sign as "where the answer goes"; minus signs as "subtract" rather than as modifiers of terms (Vlassis 2004); variables as a single mystery number.

[115] Booth et al. 2013, building on Booth 1984, Knuth, Stephens, McNeil, and Alibali 2006, and Küchemann 1978.

[116] Nicole M. McNeil and Martha W. Alibali, "Why Won't You Change Your Mind? Knowledge of Operational Patterns Hinders Learning and Performance on Equations," *Child Development* 76 (2005): 883–899. Eric J. Knuth, Ana C. Stephens, Nicole M. McNeil, and Martha W. Alibali, "Does Understanding the Equal Sign Matter? Evidence from Solving Equations," *Journal for Research in Mathematics Education* 37 (2006): 297–312.

[117] CCSSM 8.F.1.

the slope is the same between any two distinct points on a non-vertical line — the Panel's geometry foundation for linear functions.[118] A proportional relationship is the special case $b = 0$, graph through the origin. Rohrer, Dedrick, and Burgess treated "graph $y = mx + b$" and "slope from two points" as different problem kinds that students fail to associate when practice is blocked.[119] Name them as kinds. Mix them.

Wrong answers you should be able to hear

1. *2:3 treated as "two and three," so 2:3 = 4:□ becomes 4:5 or 5:6, or is a mystery.* A pair of counts, not a relationship. The child is adding or incrementing instead of scaling both parts. Ask what stays the same when both parts of a ratio are scaled. Fill a table: 2:3, 4:6, 6:9.

2. "*A negative times a negative is a positive" recited with no move and no pattern.* A slogan. Ask for $3 - (-2)$ on a line, then for $(-3) \times (-2)$ from the pattern or from two reversals. If they cannot show either, the rule has nothing to attach to.

3. *1/2 + (−1/3) = −1/5, or 1/6, or 1/5.* Adding the counts you see, with a sign stuck on. Last chapter's 2/5, now signed. Put 1/2 and −1/3 on the line. Ask which side of zero the sum lives on, and about how far, before anyone computes. The sum is 1/6, positive, closer to 0 than 1/2 is.

4. *Cross-multiply on a unit-rate that was one division.* Twelve miles in three hours, missing miles in one hour, and the child writes a proportion and crosses. The trick as a belief, not a consequence. Ask for the unit rate first. Then ask which method is cheaper here.

5. *3*x* + 2 treated as "do it," or functions introduced only as y = mx + b, or slope and intercept swapped on a graph.* The operational equal sign wearing a letter; the function concept never formed; the mixup Rohrer treated as two unassociated kinds. Ask "which values make this true?" Plot (0, *b*) first. Ask whether 0 → 0, 1 → 1, 2 → 4 is a function (yes) and whether it is linear

[118] NMAP, *Foundations for Success* (2008), geometry and measurement foundation: similar triangles as the load-bearing skill for slope and linear functions. CCSSM 8.EE: slope as unit rate; similar triangles to explain why slope is the same between any two distinct points on a non-vertical line.

[119] Doug Rohrer, Robert F. Dedrick, and Kaleena Burgess, "The Benefit of Interleaved Mathematics Practice Is Not Limited to Superficially Similar Kinds of Problems," *Psychonomic Bulletin & Review* 21 (2014): 1323–1330.

(no).

Five-minute parent warm-up

Do this on paper, no child in the room.

Minute 1. Write 2:3. Fill a table: 2 and 3, 4 and 6, 6 and 9, 1 and 1.5. Sketch a graph through the origin. Write $y = (2/3)x$. Say: "same relationship, four pictures."

Minute 2. Place −3, −1, 0, 1/2, 2 on a line. Mark $3 - (-2)$ as a move. Land on 5.

Minute 3. Write the pattern for $\times (-3)$: 2, 1, 0, −1 as the second factor. Products: −6, −3, 0, 3. Say why the last one is positive.

Minute 4. $3x + 2 = 14$. Try $x = 4$. Both sides 14. Try $x = 3$. False. The equal sign still means the same amount.

Minute 5. Write the sentence you will say: "A ratio is a relationship. A negative is a number on this line. A function gives each input exactly one output." Put the pencil down.

If you can fill the 2:3 table, show $3 - (-2)$ as a move, and say what a function is without saying $y = mx + b$, you are ready to sit down.

How to teach it this week

The math-hour box is the full shape. Scaled to this band:

Warm-up (3–5 minutes, unaided). One signed-fraction operation. One unit rate. One "is this a function?" Paper. No device.

Short model. One new idea: a ratio table next to a double number line next to $y = kx$, or a signed-number move on a line next to the written sum, or three representations of a function.

Student attempt. Three to five items. Ask, wait, hint, then model a different item. Leave the pencil in the child's hand.

One good question. "What stays the same when both parts are scaled?" "Show me the move." "Which values make both sides the same amount?" "Does each input have exactly one output?"

Mixed practice. This is Rohrer's home chapter. Day of a new skill: a short blocked set. Same week: the new skill between unlike problems — a signed-fraction sum, a unit rate, an "is this a function?", a slope from two points, a four-step linear equation. Monthly: an unannounced mixed quiz without notes. Rohrer, Dedrick, and Burgess, grade 7, unannounced test two weeks later: interleaved 72 percent versus blocked 38 percent.[120] Rohrer, Dedrick, Hartwig, and Cheung, 54 seventh-grade classes, one-month delay: 61 percent versus 38 percent.[121] Those *d* values are joint with spacing, and they are grade 7. Use them here. A purchased book that is almost entirely blocked can still be interleaved by you pulling yesterday's integers onto the same page as last month's unit rates.

Exit ticket (5 minutes, unaided). A mixed three-item check. You hold the key.

Exact wording you can say

On ratio:

"A ratio is a relationship, not two counts sitting side by side."

"What stays the same when we scale both parts?"

"Fill the table. Then the double number line. Then the graph through the origin. Same k."

"Percent is a rate per 100. 25 per 100 is the same relationship as 1 per 4."

On a cross-multiply-everything page:

"You crossed on a problem that was one division. What is the unit rate? Which method is cheaper here?"

On signed numbers:

"Show me the move. Adding a negative is a move to the left."

"$3 - (-2)$ is 3 plus the opposite of -2. Opposite of -2 is 2. Land on 5."

---

[120] Rohrer, Dedrick, and Burgess 2014: grade-7 classroom crossover, $n = 140$, unannounced test two weeks later, interleaved 72 percent versus blocked 38 percent, $d = 1.05$. Four kinds: four-step linear equation; proportion word problem; graph $y = mx + b$; slope from two points.

[121] Doug Rohrer, Robert F. Dedrick, Marissa K. Hartwig, and Chi-Ngai Cheung, "A Randomized Controlled Trial of Interleaved Mathematics Practice," *Journal of Educational Psychology* 112, no. 1 (2020): 40–52. Preregistered cluster RCT, 54 seventh-grade classes, unannounced test one month later, interleaved 61 percent versus blocked 38 percent, $d = 0.83$. Authors: interleaved assignments also space same-kind items, so d is a joint effect. Most textbooks are almost entirely blocked.

"Keep going down the pattern. What has to come next?"

On expressions and the equal sign:

"$3x + 2$ is a number that depends on x. It is not an order to 'do it.' "

"Which values make both sides the same amount? Try one. Check."

On functions:

"Each input, exactly one output. That is the whole idea."

"Show me the table, the graph, and the rule. Are they the same function?"

"Is this linear? How do you know, besides 'it looks like a line'?"

Maloney and colleagues still names the high-help temptation.[122] Prepare so you can hear a wrong k. Then the child generates. Fraulein Retanal and colleagues found, in a survey of parents of 11- to 14-year-olds — this band's ages — that a controlling-supportive homework-helping style partly mediated the parent-anxiety / child-achievement relation.[123] Preparation, then wait, is still the move.

First problem for the student

Write, exactly:

A juice mix is 2 parts juice to 3 parts water. If we use 6 parts juice, how many parts water?

Say: "A ratio is a relationship. What stays the same when both parts are scaled? You may make a table."

Wait. If the child writes 7, they added 1 to each, or treated 2:3 as "two and three" and counted on. If they write 9, they scaled both parts by 3. If they write 5, they added 2 and 3. If they stare, show a fully worked different item: 2:3 scaled to 4:□, table row 2 and 3, then 4 and 6, said as "both parts doubled; the relationship stayed." Then return to 6:□.

Later the same week, the signed diagnostic:

[122] Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock, "Intergenerational Effects of Parents' Math Anxiety on Children's Math Achievement and Anxiety," *Psychological Science* 26 (2015): 1480–1488. School-homework study.

[123] Fraulein Retanal et al., *Education Sciences* 11, no. 10 (2021): 620. Survey of parents of 11- to 14-year-olds; controlling-supportive homework-helping style partially mediated the parent-anxiety / child-achievement relation. Path analysis on a survey, not a kitchen experiment.

$1/2 + (-1/3) = \square$

Hear −1/5, 1/6-with-no-reason, 1/5. After the attempt, put 1/2 and −1/3 on the line. Estimate. Then replace with sixths.

How to fade help

First two-step equation of the form $px + q = r$: one fully worked example, then a second the child explains, then a completion missing the last step, then a conventional problem, then an incorrect example — "they added q instead of adding the opposite of q. What did they think the plus sign meant?" Booth and colleagues found that explaining incorrect examples helped conceptual understanding of exactly those two-step equations.[124] Once the child can execute, stop showing full examples.

Integer chips, if you use them, sit next to the written $-3 + 5$ and the same sum on a line, same visual field, then fade. Carbonneau, Marley, and Selig's modest average advantage for manipulatives versus symbols only was larger under guidance.[125] The conversation is the variable.

Schema-based proportion instruction: identify the schema, represent it, estimate, solve, check. Jitendra, Jon Star, Danielle Dupuis, and Michael Rodriguez, seventh grade: large gains on trained proportion items, still there at six weeks, no significant transfer.[126] Vary the cover story. A new schema will not appear for free. "Per" lives in unit rate and in percent and later in "miles per hour squared" — match the structure, not the word.

When to stop talking

[124] Booth et al., *Learning and Instruction* 25 (2013). Combined with guided practice in the Algebra I Cognitive Tutor; the fading-and-incorrect-example move is what transfers to paper.

[125] Kira J. Carbonneau, Scott C. Marley, and James P. Selig, "A Meta-Analysis of the Efficacy of Teaching Mathematics with Concrete Manipulatives," *Journal of Educational Psychology* 105, no. 2 (2013): 380–400. Overall mean d approximately 0.37 versus abstract-only, larger under guidance. Overall mean, not a grade-7 effect size.

[126] Asha K. Jitendra, Jon R. Star, Danielle Dupuis, and Michael C. Rodriguez, "Effectiveness of Schema-Based Instruction for Improving Seventh-Grade Students' Proportional Reasoning: A Randomized Experiment," *Journal of Research on Educational Effectiveness* 5 (2012). Immediate posttest $g = 1.24$; six-week retention $g = 1.27$; no significant transfer; 1,163 students, 42 classrooms. Trained proportion items. Corey Peltier and Kimberly J. Vannest, "A Meta-Analysis of Schema Instruction on the Problem-Solving Performance of Elementary School Students," *Review of Educational Research* (2017): Hedge's $g = 1.57$ on immediate elementary problem solving, mostly researcher-developed proximal measures — not a general-achievement figure.

After "what stays the same?" After "show me the move." After "is this a function?" If you hear yourself completing the cross-multiply while the child's pencil is idle, stop. Point to the table. Wait.

Geometry and data in this band are not extras. Similarity is how slope was justified and how right-triangle trigonometry will later work. Informal bivariate data — scatter plots, informal linear models, association is not causation — is the on-ramp to a later statistics course. NCTM's Focal Points warning: if algebra is offered in Grade 8, Grade 7–8 focal-point content can be moved, but not dropped.[127] A well-built project is spatial. Finding 26 still holds: it does not automatically produce a theorem.[128] Use building as a second representation of a net or a scale factor already being taught.

"Grade 8 algebra" is ambiguous. The map's Grade 8 expressions, equations, and functions is one course. A full Algebra I taken in Grade 8 is another. They are not the same. A year that only graphs $y = mx + b$ and calls it Algebra I has done neither 8.F nor Appendix A's Algebra I, if the correspondence never formed.

Practice that actually builds learning

Blocked, for a new move. The day k arrives, a short set of tables that are only "find the unit rate." The day signed addition arrives, only moves on a line. The day "each input one output" arrives, a short set of tables and graphs that are or are not functions.

Mixed, for when to use it. Rohrer's four kinds belong on the same later page: a four-step linear equation; a proportion word problem; graph $y = mx + b$; slope from two points.[129] Add a signed-fraction operation and an "is this a function?" Discrimination errors in the 2014 delayed test were rare; the authors propose that

[127] National Council of Teachers of Mathematics, *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (2006). Programs offering algebra in Grade 8 should still not omit Grade 7–8 focal-point content; it can be moved into Grade 6–7 or into high school, but not dropped.

[128] National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 26: teaching with "real-world" contexts improves performance on similar real-world items; it does not, in the small set of high-quality studies the Panel admitted, improve computation, simple word problems, or equation solving.

[129] Rohrer, Dedrick, and Burgess 2014 (n. 119–120). Four problem kinds on the delayed test: a four-step linear equation; a proportion word problem; graph $y = mx + b$; slope from two points. Discrimination errors were rare (4.4 percent); the authors propose that interleaving also strengthens the association between a problem kind and its strategy.

interleaving also strengthens the association between a problem kind and its strategy. Taylor and Rohrer 2010 is the study that most cleanly isolates interleaving from spacing; blocked-condition errors were mostly wrong-strategy errors.[130] Mix after a short block. Monthly, an unannounced mixed quiz.

Brief retrieval of facts already known. Signed-number compressions and fraction operations, mixed, after they are right untimed. Integer rules are not a chatbot job. Generate a script. Your ears do the drill. IES Recommendation 6 still applies: brief timing on already-known material.[131] Boaler's object is still a different object.

One incorrect example to diagnose. A page on which every item was cross-multiplied, including a unit rate. $1/2 + (-1/3) = -1/5$. A graph of $y = 2x + 1$ plotted with slope 1 and intercept 2. $3x + 2$ "done" into a single number with no x. The child names the misconception. You do not announce a talent verdict.

Kitchen, money, and making. A 20 percent tip can be a second representation of $20/100 = 1/5$. A scale drawing of a room can be a second representation of k. Finding 26: those contexts help similar contexts. A child who can compute a tip and cannot write a unit rate has performed, not learned.[132] Shopping does not replace the table. Cooking does not replace the line.

Schema work still rejects keywords. Additive: combine, change, compare. Multiplicative: equal groups, comparison, ratio, rate, percent, proportion. Introduce one type at a time. Complete story first. Then one unknown. Vary the cover story.

A weekly shape: new skill blocked. Same week, mixed retrieval. Later this week and next, the new skill between unlike problems. Monthly, unaided mixed quiz. You impose that shape on the book you already have.

---

[130] Kelli Taylor and Doug Rohrer, "The Effects of Interleaved Practice," *Applied Cognitive Psychology* 24 (2010): 837–848. Both conditions spaced; $d = 1.23$; blocked-condition errors were mostly wrong-strategy errors. A different design from the 2014 classroom crossover.

[131] Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*, WWC 2021006 (IES, 2021), Recommendation 6: regularly include timed activities as one way to build fluency, on already-taught material, brief, mixed. Jo Boaler, "Timed Tests and the Development of Math Anxiety," *Education Week* / Stanford GSE (2012), and *Fluency Without Fear* (Youcubed working paper, 2015), attack a different object: high-stakes early speed tests.

[132] NMAP Finding 26 (n. 128). Nicholas C. Soderstrom and Robert A. Bjork, "Learning Versus Performance: An Integrative Review," *Perspectives on Psychological Science* 10, no. 2 (2015): current performance is "often an unreliable index" of learning.

For the student

A **ratio** is a relationship. 2:3 is not “two and three.” It is a pairing that stays the same when both parts are scaled: 2:3, 4:6, 6:9.

A **negative** is a number on the same line as 3 and 1/2, to the left of zero. Rules are nicknames for moves. $3 - (-2)$ is a jump of 2 to the right from 3, landing on 5.

A **function** gives each input exactly one output. Some functions make a line. Some do not. The idea is the pairing, not the slope-intercept shape.

The equal sign still means the **same amount**, including when a letter is in the sentence.

Tiny worked example

Juice to water is 2:3. We have 4 parts juice. Water?

Both parts doubled. 2 became 4, so 3 becomes 6. Table:

juice: 2, 4 water: 3, 6

Same relationship. Water is 6.

Two tries

1. Same 2:3 mix. 6 parts juice. How many parts water? (Table or scale — your choice.)
2. Place −3, 0, and 2 on a number line. Then compute $3 - (-2)$. Show the move, then write the number.

Explain it back

Tell someone at the table, in your own words, what a ratio is. Then say why a negative times a negative is a positive — using a pattern *or* a reverse-direction picture, not only the slogan. Then say what a function is without using the sentence $y = mx + b$.

Challenge

Is this a function? Input 0 gives 0, input 1 gives 1, input 2 gives 4, input 3 gives 9. If it is a function, is it linear? Sketch. Then find the unit rate for 12 miles in 3 hours, *without* cross-multiplying, and say why that method is enough.

You are allowed to struggle. Draw the table. Draw the line. Write. If you get stuck, ask for a hint — not the answer.

Shopping and cooking can show a percent twice. They do not replace the table you just built.

If it isn't clicking

1. Every proportion is cross-multiplied, including the one-step unit rate, and 2:3 still sounds like "two and three."

The relationship never formed. Next move: tables and double number lines only, for a week. Scale up, scale down, find k, graph through the origin. Hide the cross. Slow down formal proportions until the table is hearable. Go ahead to $y = kx$ and to percent-as-rate-per-100 once 2:3 = 4:6 is obvious and the child can say what stayed the same. A tutor who will sit with a table, not a cross-multiply chant, is useful if the pair-of-counts reading is still the default after that week.

2. Signed-number slogans without a line; $1/2 + (-1/3)$ is a counts-you-see error with a sign stuck on.

The numbers never landed. Next move: the line every day. Opposites. Absolute value as distance. Addition as a move. $3 - (-2)$ as a move. Pattern for multiplication before the slogan. Signed fractions on the same line as last chapter's unsigned ones. Slow down two-step equations with negative coefficients until the moves are hearable. Go ahead once $-3/4$ and $2/3$ can be placed and $1/2 + (-1/3)$ is estimated before it is computed. A tutor is reasonable if last chapter's 2/5 is still in the room *and* the sign work is collapsing; that is an unfinished gate, not a reason to buy Algebra I.

3. $3x + 2$ is "do it"; functions are only y = mx + b; slope and intercept swap.

The operational equal sign never retired, and 8.F never opened. Next move: true/false with letters. Substitute to check. "Which values make this true?" including always-true and never-true: $2x + 3 = 2x + 5$ is never true. Three representations of a function in every sitting, including one nonlinear. Plot $(0, b)$ first. Mix "graph this" with "slope from two points." Slow down Algebra I until substitution-as-check is a habit and "each input one output" can be said without a slope. A tutor who will refuse to start the high-school book is more useful than a tutor who will "push through the algebra."

When to slow down. Last chapter's gate unfinished: 3/4 has no home on a line; 1/2 + 1/3 is still 2/5. Integer operations fragile. Equal sign still "here comes the answer." Similarity and Grade 8 data about to be dropped because an Algebra I book is on the shelf.

When to go ahead. Ratios as relationships, unit rate, *k*, graph through the origin, percent as rate per 100. Signed fractions operated, on the line. Expressions evaluated; two-step equations solved *and* checked by substitution. Function as correspondence; linear versus nonlinear; slope as unit rate from similar triangles. Then Algebra I has something to attach to. Compacted 7–8, Appendix A's model, only if those gates are open.

When to get a human tutor. Honest daily work on the table, the line, and substitution-as-check, the same diagnosis still in the room, and the hour a fight. NCES 2016: 23 percent of homeschool families used any tutor; 31 percent any local group or co-op.[133] A tutor is a release valve. Video curricula exist for the same reason. Keep yourself as the person who can still hear a wrong *k* and a slogan with no move. When you are the ceiling, a co-op or a later dual-enrollment course is a next step, not a failure of the sitting.

For a child already behind on this topic, explicit systematic instruction has consistently positive effects.[134] Productive failure is generation-then-instruction when priors exist.[135] A child missing signed-fraction operations needs a model first. A child who has the operations and is meeting "which values make this identity always true?" can generate, then receive the instruction.

[133] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey* (NCES 2020-001): 23 percent any tutor; 31 percent any local homeschool group or co-op.

[134] NMAP, *Foundations for Success* (2008), Finding 27: explicit instruction (clear models, extensive practice, think-alouds, extensive feedback) has consistently positive effects for students with mathematical difficulties. Fuchs et al. 2021, Recommendation 1 (systematic instruction, strong evidence, 43 studies). Finding 27 notes that this does not mean all instruction should be explicit.

[135] Manu Kapur, "Productive Failure in Learning Math," *Cognitive Science* 38, no. 5 (2014): 1008–1022. Generation-then-instruction on a new concept when priors exist. A different comparison from Finding 27.

Tools, including AI

Optional helpers for you. The full rules live in the math-hour box. This section stays short.

The child attempts first. You hold the key. A tool may explain to *you* why a negative times a negative is a positive (pattern and directed-magnitude), why a ratio is a relationship, or why 8.F is not "$y = mx + b$." It may make extra proportion items with a parent key, shuffle a mixed page of Rohrer's kinds, write a hint script that names "ratio as a pair of counts," or diagnose a page already produced.

A ratio table is a facts-adjacent object. Language models hallucinate arithmetic. Bastani's unguarded tutor raised practice scores and cut the unaided exam, and it made arithmetic errors even when the method was right.[136] You check the table.

Paste the pre-algebra worksheet, photo-to-key, "what's the answer," and an unsupervised chatbot during the attempt stay out. "Finish your pre-algebra with a solver so we can go to soccer" is the pattern with a large-sample causal harm estimate. Integer rules are not a chatbot job. Generate a script. Your ears do the drill. The math-hour box is the rest.

What "done enough" looks like

Placement is by skill, not birthday. Dimensions 6 is not pre-algebra. Saxon 8/7 is a skill name; transcript titles a stranger can read come later in this book. Math-U-See Pre-Algebra is a negatives-and-unknowns level. Beast Academy Level 5 is not "fifth grade." Calendar-grade-8 Algebra I is not identical to the map's Grade 8 functions. Place by what the child can hear.

Checklist before moving on

- A ratio is a relationship. 2:3 = 4:6 is obvious in a table, on a double number line, and as $y = (2/3)x$ through the origin. Unit rate is one division when that

[136] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. Unguarded tutor raised assisted practice and cut the unaided exam; hint-only tutor did not hurt the exam. Arithmetic errors even when the method was right. Not a homeschool study. The pattern is the sitting condition.

is cheaper. Cross-multiply is a consequence the child can explain, not the first move on every item.

- Percent is a rate per 100. Tax, tip, markup, and percent increase sit on that rate, not on a chant alone.
- Fraction ÷ fraction has a measurement meaning; invert-and-multiply is a compression of that meaning.
- Signed rationals live on the same line as 3/4. Opposites, absolute value, four quadrants. Addition is a move. Subtraction is adding the opposite. Multiplication has a pattern justification and a reverse-direction justification. $1/2 + (-1/3)$ is not $-1/5$, and the child can estimate first.
- An expression is a number that depends on x. Solving asks which values make both sides the same amount, and substitution checks. Always-true and never-true equations are allowed to exist.
- A function assigns to each input exactly one output. Table, graph, rule, verbal, in one sitting. Linear versus nonlinear. $y = mx + b$ is one family, m a unit rate, b an initial value, slope justified by similar triangles.
- Congruence via rigid motions; similarity via dilations; Pythagorean theorem as a statement about right triangles, including distance on the grid. Informal scatter plots: association is not a function until someone specifies a rule, and it is not causation.
- The equal sign still means the same amount, including inside $-2x + 5 = 11$.
- Shopping, cooking, and building were second pictures. The table, the line, the written equation, and the unaided mixed page did the work.
- You can hear a pair-of-counts ratio, a slogan with no move, a signed 2/5, a cross-on-everything page, and a function that is only a slope-intercept shape — and ask a good question without taking the pencil.

If most of that list is true, Algebra I is next, even if the birthday says otherwise. If the birthday says “Algebra I this year” and 3/4 still has no home, or 2:3 is still two counts, or −3 is still a slogan, stay. Diagnose fractions including signed, integer operations, and the equal sign before buying the book. The 2007 survey of 743 Algebra I teachers named rational numbers as the top incoming gap.[137] A child who can operate on signed rationals, see a ratio as a relationship, treat “=” as a relation, and assign to each input exactly one output is ready. A child who cannot

[137] NMAP, *Foundations for Success* (2008), nationally representative survey of Algebra I teachers, 743 teachers, 2007: poorest incoming preparation in “rational numbers and operations involving fractions and decimals,” alongside word problems and study habits. 2008 data, not a 2026 classroom census.

is not ready, regardless of calendar grade, regardless of a publisher placement test, regardless of a chatbot that can print the next chapter's worked examples while the child watches.

A path through mathematics is the promise. A percentile is not.

Chapter 4

Algebra I

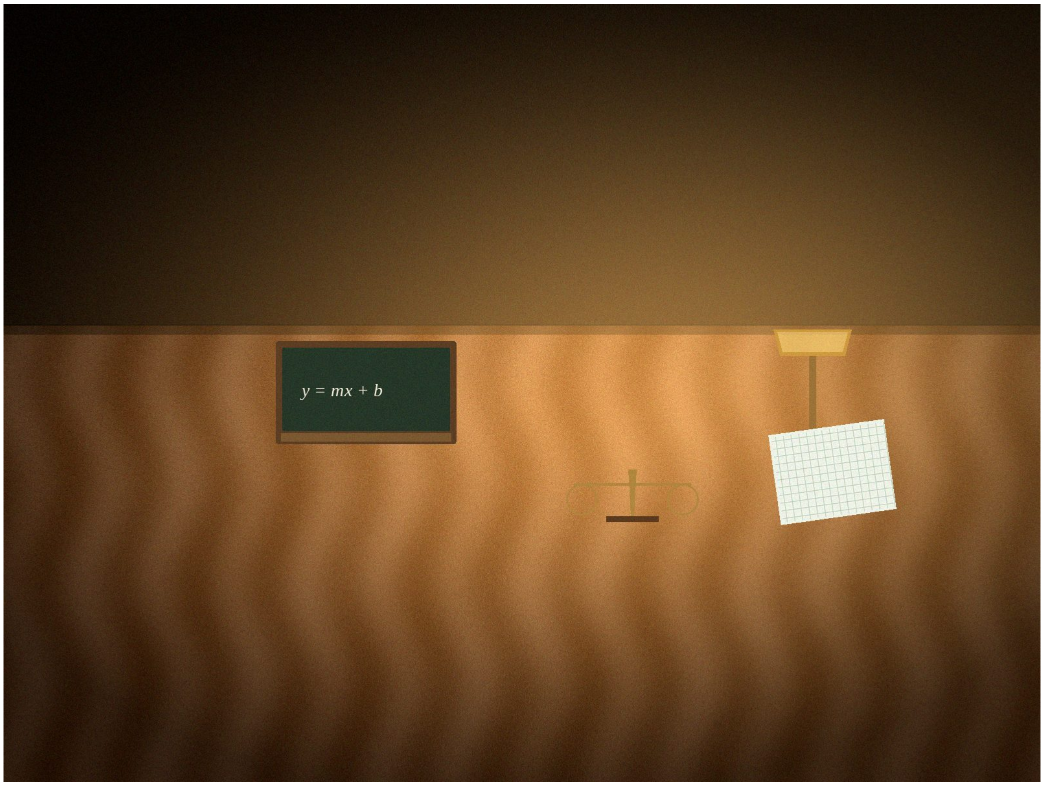

Figure 5: A chalkboard with y=mx+b, a balance scale, and a sheet of graph paper.

Why this matters

Algebra I is the year letters start doing the work numbers already knew how to do. It is the year a child learns that an equation is a claim about two amounts, that a function is a rule that takes each allowed input to exactly one output, and that three families of growth — linear, exponential, quadratic — are not the same story wearing different clothes.

That is worth the struggle because almost everything later hangs on it. Geometry will ask for coordinate proof and for completing the square when a circle's equation arrives. Algebra II will ask for polynomials beyond quadratic, for logarithms, and for trigonometry on the unit circle. Calculus is a new operation on functions the student already knows. A shaky function concept in this course becomes symbol soup later. A solid one becomes a place a derivative can attach.

The course is not a birthday. Achieve, working with the Common Core writing team, published a model for Algebra I that says the fundamental purpose is to formalize and extend the mathematics of the middle grades, and that this is a more ambitious first high-school algebra than has generally been offered.[138] Typical U.S. placement is Grade 9 on a non-compacted path, Grade 8 on a compacted path, sometimes Grade 7. Those are calendar facts, not developmental laws. The National Mathematics Advisory Panel's Finding 15 is still the useful refusal: claims that children of a given age cannot learn certain content because they are "too young" or "not ready" have consistently been shown to be wrong.[139] Accelerate a child who has the gates. Wait with a child who does not, even if the neighborhood ninth-graders have already started.

At home, Algebra I is common. In the 2016 federal parent survey, 79 percent of high-school-level homeschoolers had ever been taught basic algebra; 41 percent

[138] Achieve / Common Core writing team, *Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards*, Oregon Department of Education copy, https://www.oregon.gov/ode/educator-resources/standards/mathematics/Documents/math-appendix-a-model-course-pathways.pdf, accessed 30 August 2026. Algebra I: "The fundamental purpose of this course is to formalize and extend the mathematics that students learned in the middle grades. Because it is built on the middle grades standards, this is a more ambitious version of Algebra I than has generally been offered." Pathways and courses are models, not mandates.

[139] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (U.S. Department of Education, 2008), ERIC ED500486, Finding 15.

of the 6–8 band were being taught Algebra I that year.[140] Geometry and Algebra II are less universal. Calculus at home is a minority. A family that treats this course as rare has left the table. A family that treats a calendar-grade-8 Algebra I as identical to Grade 8 expressions, equations, and functions has left it the other way. Those are not the same course.

What this course is, in one page: relationships between quantities and reasoning with equations; linear and exponential functions, notation, domain and range, systems; descriptive statistics — shape, center, spread, two-way tables, a linear fit, correlation versus causation; quadratic expressions and equations, including factoring, completing the square, and the quadratic formula with real solutions; quadratic functions compared with linear and exponential; a first look at absolute value, step, and piecewise-defined functions.[141] Complex solutions of quadratics are recognized. They wait for Algebra II to be developed.

What this course is not: a 180-day worksheet dump, a promise of a score, or the whole of school algebra. Polynomials beyond quadratic, complex numbers, logarithms, and trigonometry wait. A year that graphs $y = mx + b$, solves two-step equations with whole-number coefficients, and never reaches a function concept, an exponential table, or a quadratic with real solutions is algebra in name. Name the year by what was taught. Transcript it as **Algebra I**, not as a publisher spine a stranger cannot map.

Before you buy the book, run the door. The diagnostic is content, not a birthday. Can the student operate on rationals, including signed fractions? See structure in an expression? Treat the equal sign as a relation, now inside a multi-step equation? A student who cannot is not ready, regardless of calendar grade. The rest of this chapter teaches the course. This page names why the course is worth a hard year, and why the door is the first teaching move.

[140] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey*, NCES 2020-001 (Washington, DC: NCES, 2019), Tables 8–9. Algebra I taught that year to 41 percent of the 6–8 homeschool band and 33 percent of 9–12; basic algebra (Algebra I) ever taught to 79 percent of high-school-level homeschoolers. The 2023 First Look (NCES 2024-113) did not republish those subject tables.

[141] Appendix A (n. 138), Algebra I five critical areas.

For the parent: understand it yourself

You do not need to be a mathematician. You do need to understand today's idea well enough to hear $(x + 2)(x + 3)$ written as $x^2 + 5$ as a distribution miss, and $3 + 4 + 5 = 3 + \square$ written as 15 as an equal-sign miss, not as cute slips.

Everyday picture. Think of a balanced scale. Whatever you do to one pan, you do to the other, because the two pans already name the same amount. "Equals" is not a button that means "compute now, write the answer on the right." It is the claim that the two sides are the same.

Precise picture. Algebra is generalized arithmetic. The letters stand for numbers, and the properties you already know for numbers still hold.[142] $3(x + 2)$ is three times a sum, the same distributive property the child used on $3(10 + 2) = 36$, now with a letter. A variable is not a single mystery number. $3x + 2$, when x is 4, is 14. When x is -1, it is -1. The expression is a number that depends on x. Function notation later this year — $f(x) = 3x + 2$ — is the same idea with a name for the correspondence. Domain is the set of inputs the correspondence will accept. Range is the set of outputs it produces. $f(x)$ is not f times x. It is the output of f at input x.

Solving is asking which numbers, substituted for the letter, make the two sides name the same quantity. Write $-2x + 5 = 11$. Try $x = -3$: $-2(-3) + 5 = 6 + 5 = 11$. True. Try $x = 2$: $-4 + 5 = 1$. $1 = 11$ is false. The inverse steps — add $2x$, subtract 5, divide by -2, or add the opposite of 5, then divide — are a compression of that search, not a dance. Infinitely many solutions and no solution are the first time an equation is a claim that can be always true or never true, not only sometimes true. $2x + 3 = 2x + 5$ is never true. $2x + 3 = 2(x + 1) + 1$ is always true. Substituting still decides.

Two justifications for $(x + 2)(x + 3) = x^2 + 5x + 6$, both mathematics.

The area-model justification. A rectangle with sides $x + 2$ and $x + 3$ has area $(x + 2)(x + 3)$. Split each side: an x-by-x square, an x-by-3 rectangle, a 2-by-x rectangle,

[142] Algebra as generalized arithmetic: letters stand for numbers and the properties already known for numbers still hold. Julie L. Booth, Karin E. Lange, Kenneth R. Koedinger, and Kristie J. Newton, "Using Example Problems to Improve Student Learning in Algebra: Differentiating between Correct and Incorrect Examples," *Learning and Instruction* 25 (2013): 24–34; Eric J. Knuth, Ana C. Stephens, Nicole M. McNeil, and Martha W. Alibali, "Does Understanding the Equal Sign Matter? Evidence from Solving Equations," *Journal for Research in Mathematics Education* 37, no. 4 (2006): 297–312.

a 2-by-3 rectangle. Areas: x^2, $3x$, $2x$, 6. Sum: $x^2 + 5x + 6$.

The distributive-property justification. $(x + 2)(x + 3) = (x + 2)x + (x + 2)\cdot 3 = x^2 + 2x + 3x + 6 = x^2 + 5x + 6$. The $x + 2$ hits both terms of $x + 3$, because multiplication distributes over addition. FOIL — first, outer, inner, last — is a mnemonic for a special case of that property applied twice. It fails the moment the product has three terms on one side.

Factoring is the same product read the other way. Completing the square is a different rewriting: $x^2 + 6x + 5 = (x + 3)^2 - 4$, because $x^2 + 6x$ wants a 9 to be a square, and 5 is 9 minus 4. The quadratic formula is completing the square done once on $ax^2 + bx + c = 0$ and then used as a compression. Three rewritings of the same quadratic, not three franchises.

Three function families are the spine. A linear function has a constant rate of change. An exponential function has a constant multiplicative rate. A quadratic has a constant second difference, a vertex, and a graph that is a parabola. Three representations in one sitting: table, graph, formula, and a verbal description. If the table says $0 \to 3$, $1 \to 5$, $2 \to 7$, $3 \to 9$, the first differences are constant 2, the rule is "start at 3, add 2 each time," the graph is a line with slope 2 and intercept 3, and the equation is $y = 2x + 3$. If the table says $0 \to 3$, $1 \to 6$, $2 \to 12$, $3 \to 24$, the ratios are constant 2, the rule is "start at 3, multiply by 2 each time," and the equation is $y = 3 \cdot 2$^x. If the table says $0 \to 0$, $1 \to 1$, $2 \to 4$, $3 \to 9$, the first differences are 1, 3, 5 and the second differences are constant 2, the rule is "square," and the equation is $y = x^2$.

Slope m is the rate of change, grounded in similar triangles.[143] Intercept b is the output when the input is zero. A proportional relationship is the special case $b = 0$. Average rate of change of f from a to b is $[f(b) - f(a)] / (b - a)$. That is slope between two points on the graph of f, whether or not f is linear. For a linear function it is constant. For a quadratic it is not. That difference is how the families are told apart.

$y = f(x) + k$ shifts the graph of f up by k if k is positive. $y = f(x - h)$ shifts the graph of f right by h if h is positive. The minus in $x - h$ is the one the child will get wrong, because "minus means left." It does not. $f(x - 2)$ takes the same output when the input is two larger, which is a shift right.

[143] NMAP, *Foundations for Success* (2008), Critical Foundations: similar triangles, because slope and linear functions "depend logically" on them.

Systems of linear equations are three representations again: a graph of two lines and their intersection; an algebraic elimination or substitution; an inspection of structure. Two lines may meet at one point (one solution), be parallel (no solution), or be the same line (infinitely many). Substituting still decides. Linear inequalities extend the same relation: the solution is a set, a ray on a line or a half-plane, not a single number. A line and a parabola may meet at zero, one, or two real points.

Arithmetic and geometric sequences are the discrete versions of the first two families. An arithmetic sequence 3, 5, 7, 9, … has first term 3 and common difference 2; the nth term is $3 + (n - 1)\cdot 2$, which is linear in n. A geometric sequence 3, 6, 12, 24, … has first term 3 and common ratio 2; the nth term is 3 · 2^(n − 1), which is exponential in n. Ask which family the sequence belongs to, then write the formula. A list without a family is a chant.

$|x|$ is distance from zero, now as a function whose graph is a V. A step function is constant on intervals and jumps. A piecewise function is different rules on different parts of the domain. Three representations in one sitting. Functions are not only $y = mx + b$.

Descriptive statistics already live here: shape, center, spread; two-way tables; scatter plots; a linear fit; residuals; correlation versus causation.[144] A residual is the difference between an observed y and the y the line predicts. Correlation r is a number between −1 and 1 that measures the strength and direction of a linear association. $r = 0.9$ does not mean X causes Y.

Wrong answers you should be able to hear

1. The operational equal sign. The child writes 3 + 4 + 5 = 3 + □ as 15, or treats $11 = -2x + 5$ as "backwards," or says "equals means the total." Nicole M. McNeil and Martha W. Alibali named three entrenched operational patterns: perform all given operations on all given numbers; the "operations = answer" visual structure; "=" means "the total."[145] Eric J. Knuth and colleagues connected a relational view

[144] Common Core State Standards for Mathematics, S.ID.1–3, 5–9, as placed in Appendix A's Algebra I course. College Board, *AP Statistics Course and Exam Description*, Effective Fall 2026: prerequisite is successful completion of a first-year algebra course. This course is that prerequisite; these clusters are the on-ramp, not the AP course.

[145] Nicole M. McNeil and Martha W. Alibali, "Why Won't You Change Your Mind? Knowledge of Operational Patterns Hinders Learning of Arithmetic Equations," *Child Development* 76, no. 4 (2005): 883–899.

of the equal sign to later algebra.[146] Those patterns do not retire because the cover says Algebra I.

What it usually means. The child still hears "=" as "compute now, the answer goes on the right," not as "these two sides name the same amount."

What to say. "Equals means the two sides are the same amount. Which number in the box makes that true? You may not add numbers that sit on only one side."

2. $(x + 2)(x + 3)$ written as $x^2 + 5$. Or as $x^2 + 6$, with the insides added and the x term missing.

What it usually means. The child did not distribute. The binomials were treated as two numbers to be added, or the product of the constants was written as the only extra term.

What to say. "The 2 has to multiply besides the x. Draw the rectangle. Four areas. Then write the four terms."

3. $3 + -2$ treated as "three plus minus two, so subtract, I guess." Or $-2x$ heard only as a subtraction waiting to happen.

What it usually means. Minus is being used only as "subtract," not as a modifier of a term. Joëlle Vlassis, as cited by Julie L. Booth and colleagues, is the source for this misconception.[147]

What to say. "Write $3 + (-2)$. That is a sum of a number and a signed number. $-2x$ is a term whose coefficient is -2."

4. A variable treated as one mystery number. The child can "solve for x" in $2x + 1 = 7$ and cannot say what $2x + 1$ is when x is -1, or cannot accept that x can be many values in $f(x) = 2x + 1$.

What it usually means. Letters still mean "the unknown we find once," not "a symbol that can take many values."

What to say. "When x is 4, this expression is 9. When x is -1, it is -1. The letter can take many values. The expression is the number it becomes."

[146] Knuth et al., "Does Understanding the Equal Sign Matter?" (2006).

[147] Booth et al., "Using Example Problems" (2013). Misconception list includes minus signs treated as "subtract" rather than as modifiers of terms; Joëlle Vlassis, "Making Sense of Signs of Subtraction and the Equal Sign in Algebra" (2004), as cited by Booth et al.

5. "If the correlation is 0.9, *X* causes *Y*." Or every kitchen story treated as $y = mx + b$.

What it usually means. Association has been upgraded to cause, or a context has been upgraded to a family. The National Mathematics Advisory Panel's Finding 26 is still the caution: teaching with real-world contexts improves performance on similar real-world items; it does not, in the small set of high-quality studies the Panel admitted, improve computation, simple word problems, or equation solving.[148] A doubled recipe is a scaling if both quantities scale. Shopping will not teach completing the square.

What to say. "The number r tells how tightly the points hug a line, and in which direction. It does not tell us that one variable makes the other happen. Name the family first. Then, if you want, tell the story."

Five-minute parent warm-up

Before the lesson, on paper, no child in the room:

1. Expand $(x + 2)(x + 3)$ two ways: the rectangle and the distributive property. Circle the term a child who writes $x^2 + 5$ is missing.
2. Solve $-2x + 5 = 11$ by substituting two trial values, one true and one false, then by inverse steps. Check by substituting back.
3. Look at one table. Decide whether it is linear, exponential, or quadratic, and say how you know.

If those three are fluent for you, you can hear a wrong inverse. That is the job. The child still holds the pencil.

How to teach it this week

Run the door before the first Algebra I lesson. Three questions, not a publisher placement test.

Fractions, including signed. Can the child add, subtract, multiply, and divide signed fractions? Place $-3/4$ and $2/3$ on the same number line? Treat $1/2 + (-1/3)$ as a magnitude-and-sign problem, not as adding the counts you see? The Panel

[148] NMAP, *Foundations for Success* (2008), Finding 26.

named three Critical Foundations of Algebra: fluency with whole numbers; fluency with fractions, including decimals, percents, and negative fractions; and particular aspects of geometry and measurement, especially similar triangles, because slope and linear functions depend logically on them. Of these, "knowledge of fractions is the most important foundational skill not currently developed among American students."[149] A 2007 nationally representative survey of 743 Algebra I teachers, commissioned by the Panel, rated incoming preparation as poorest in rational numbers and operations involving fractions and decimals.[150] Robert S. Siegler and colleagues found that elementary fraction and division knowledge uniquely predicted high-school algebra five or six years later, after other mathematical knowledge, general intellectual ability or working memory, and family income and education.[151] That is a prediction finding, not a classroom experiment proving that one fractions curriculum causes algebra success. It is still a reason the gate is load-bearing.

Signed numbers. Can the child treat $3 + (-2)$ as a sum, not as a chant? Write $(-3) \times (-2) = 6$ as a pattern that continues whole-number arithmetic, not as a slogan? Parentheses around a negative, early and often.

The equal sign as a relation. Can the child treat "=" as "names the same quantity," now inside $-2x + 5 = 11$? A child who can mechanically isolate x and cannot substitute back to check has a procedure and not the relation.

If those three are fragile, this book's middle-grades chapters are the repair, not a pep talk. Then return.

When the gates are open, here is the shape of a sitting. Warm-up. Short model. Student attempt. One good question. Mixed practice. Exit ticket.

Warm-up (3–5 minutes, unaided). One signed-fraction two-step equation. One

[149] NMAP, *Foundations for Success* (2008), Executive Summary findings 4 and 12, and Chapter 4. Three Critical Foundations of Algebra. Quoted clause: "knowledge of fractions is the most important foundational skill not currently developed among American students."

[150] NMAP, *Foundations for Success* (2008), nationally representative survey of Algebra I teachers, 743 teachers, 2007. These are 2008 data, not a 2026 classroom census.

[151] Robert S. Siegler, Greg J. Duncan, Pamela E. Davis-Kean, Kathryn Duckworth, Amy Claessens, Mimi Engel, Maria Ines Susperreguy, and Meichu Chen, "Early Predictors of High School Mathematics Achievement," *Psychological Science* 23, no. 7 (2012): 691–697. British Cohort Study $N = 3{,}677$; U.S. PSID-CDS $N = 599$. A 1-SD increase in age-10 fractions knowledge uniquely associated with a 0.15-SD (UK) / 0.17-SD (US) increase in high-school algebra after controls. Prediction finding, not a fractions-curriculum experiment.

"which family is this table?" One expansion. Yesterday's work, not tonight's new move.

Short model (5–8 minutes). One fully worked example on paper. John Sweller and Graham Cooper showed that for novices, studying worked solutions outperformed unguided problem solving, because search consumes working memory that could have been used to extract the schema.[152] Fade: a second example the child must explain, then a completion problem missing the last step, then a conventional problem.[153] Once the child can execute, stop showing full examples. The same full example that helps a novice can become redundant later.

Include at least one incorrect example per new procedure. Booth, Lange, Koedinger, and Newton found that explaining worked examples during guided practice improved conceptual understanding, and that incorrect examples, alone or mixed with correct ones, were especially useful for conceptions.[154] Ask "what did they think this symbol meant?" "They forgot the rule" is not a diagnosis. "They treated the equal sign as the place the answer goes, so they added everything they saw" is.

Exact wording you can steal.

On a system: "Two lines. One meeting point, no meeting point, or the same line. Substitute your pair back into *both* equations. A pair that works in one and fails in the other is not a solution."

On a sequence: "Is the next term made by adding, or by multiplying? That is the family. Now write the nth term."

On statistics: "Ask a question that has variability. Plot. Name the shape. Fit a line only if the scatter suggests one. The line is a model, not a cause."

On the equal sign: "Equals means the two sides name the same amount. Which numbers, put in for x, make that true?"

[152] John Sweller and Graham A. Cooper, "The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra," *Cognition and Instruction* 2, no. 1 (1985): 59–89.

[153] Alexander Renkl and Robert K. Atkinson, guidance fading / backward fading, as synthesized from Renkl and Atkinson 2003; Renkl, Atkinson, Maier, and Staley 2002; Atkinson, Derry, Renkl, and Wortham 2000. Slava Kalyuga, Paul Ayres, John Chandler, and John Sweller, "The Expertise Reversal Effect," *Educational Psychologist* 38, no. 1 (2003): 23–31.

[154] Booth et al., "Using Example Problems" (2013). Combined with guided practice in the Algebra I Cognitive Tutor in that paper — a school study, not a homeschool result.

On distribution: "The 3 hits both terms, because multiplication distributes over addition. Say that, then write the two terms."

On a first two-step equation: "You wrote $-2x + 5 = 11$ and added 5. What does the plus 5 already mean on the left? What would keep the two sides the same?"

On a transformation: "$f(x - 2)$ is the output when the input is two larger. That moves the graph right, not left. Check with a table: when x is 2, the inside is 0."

On a family: "Are the first differences constant, the ratios constant, or the second differences constant? Name the family before you write the formula."

First problem for the student, after the model. Solve $-2x + 5 = 11$. Substitute $x = -3$ and $x = 2$ first. Then write the inverse steps. Substitute back. The child writes. You wait.

One good question. After the attempt: "Why did you add the opposite of 5, rather than add 5?" The child generates the why. You do not narrate it.

Student attempt, then mixed practice. Two or three items of the new type, unaided, after the fade. Then mix: yesterday's linear system, last month's exponential table, a quadratic to factor, one "which family?," one correlation-versus-causation item. Doug Rohrer and colleagues showed, in grade 7 classrooms, that mixing problem kinds that share a surface and differ in strategy beats blocking when the test is delayed.[155] Those trials were not Algebra I. The practice of mixing types is still the move. A purchased book that is almost entirely blocked can still be interleaved by you pulling yesterday's system onto the same page as last month's table.

Exit ticket (5 minutes, unaided). A mixed three-item check without notes and without a tool. One two-step with a signed-fraction coefficient. One expansion or factoring. One "which family?" You grade it. Current performance on a page with the model at the elbow is often an unreliable index of learning.[156] The unaided ticket is the index you want.

How to fade help. Complete example → child explains a second → last step blank

[155] Doug Rohrer, Robert F. Dedrick, and Kaleena Burgess, "The Benefit of Interleaved Mathematics Practice Is Not Limited to Superficially Similar Kinds of Problems," *Psychonomic Bulletin & Review* 21 (2014): 1323–1330. Grade-7 classroom crossover. The practice of mixing types is used here; the coefficient stays with that trial.

[156] Nicholas C. Soderstrom and Robert A. Bjork, "Learning Versus Performance: An Integrative Review," *Perspectives on Psychological Science* 10, no. 2 (2015).

→ conventional problem → incorrect example named → child generates. When the child is accurate without the rectangle, put the rectangle away. Bring it back when $(x + 2)(x + 3)$ becomes $x^2 + 5$ again.

When to stop talking. After one model and one question. Struggle before rescue. Ask, wait, hint, then model again if needed. The pencil stays with the child. A parent who last sat this course years ago can still hear a wrong expansion. Hearing is not the same as taking over the page.

Schema instruction for word problems: a linear comparison, a mixture, a motion, a work problem, a system of two constraints, an exponential growth, a projectile that is a quadratic in time. Introduce one type at a time. Represent it. Estimate. Solve. Check. Vary the cover story so the child is matching structure, not nouns. "Per" lives in unit rate and in percent and later in other units. The structure, not the word.[157]

Practice that actually builds learning

Understanding and practice grow together. Practice a new rewriting this week, with a picture and a why. FOIL without the distributive property is a mnemonic with no home.

Blocked, for a new move. The day a child first completes the square, a short blocked set is honest: several items that all want the same move, so execution can form. Same for a first system by substitution, a first expansion, a first "read the slope and intercept from a graph."

Mixed, for when to use it. Later the same week, and next week, the new skill reappears between unlike problems. A page that is only "solve the quadratic by factoring" teaches execution. A page that mixes "is this linear, exponential, or quadratic?," "solve by factoring," "solve by formula," "fit a line to these four points," and "does this system have one solution, none, or infinitely many?" teaches choosing.

Brief retrieval of facts already known. Signed-fraction operations, integer facts,

[157] Asha K. Jitendra, Jon R. Star, Danielle N. Dupuis, and Michael C. Rodriguez, schema-based instruction RCT, 2012, seventh-grade proportion problems. The method, not a promise of that trial's effect on Algebra I items. IES / Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics*, WWC 2021006 (2021), Recommendation 5: keyword strategies fail.

and linear-equation inverse steps the child can already get right untimed. Time only those. Fluency is quick, flexible, accurate recall — not a race and not optional. A short mixed oral at the start of the hour is enough: "What is $(-3) \times (-2)$? Place $-3/4$. What is $1/2 + (-1/3)$?"

One incorrect example to diagnose. Show a completed square that added b instead of $(b/2)^2$. Or an expansion $(x + 2)(x + 3) = x^2 + 5$. Or $3 + 4 + 5 = 3 + \square$ filled with 15. Ask what they thought the symbol meant. Then an isomorphic item the child does unaided.

Compare methods after execution exists. After the child can factor $x^2 + 5x + 6 = 0$, put completing the square next to it, then the formula on $x^2 + x + 1 = 0$, and notice the discriminant is negative — real solutions do not exist; complex solutions are recognized, not developed. "Which is better for this problem, and why?" Flexibility is a third outcome, not a luxury.[158]

Kitchen, money, and making can motivate or give a second picture. They do not replace the number line, the written equation, or the unaided problem. Compound interest is a legitimate exponential model. It will not, by itself, teach 2^x. The table of ratios, the graph that is not a line, and the comparison with $2x$ will. Use the context as a second representation of a growth the child is already being taught as a family.

A child who can compute a 20 percent tip and cannot write $y = 0.20x$ as a linear function through the origin, or who writes $y = 0.20x + 20$ because "there is always a 20," has performed, not modeled. Name the family.

Monthly: an unannounced mixed quiz the child does without notes. That is the retrieval the year needs, not a stack of identical worksheets the night before.

For the student

This course is the year letters stand for numbers, and the properties you already know still hold. $3(x + 2)$ is three times a sum, the same move as $3(10 + 2)$. The equal sign means the two sides name the same amount. It does not mean "here

[158] Michael Schneider, Bethany Rittle-Johnson, and Jon R. Star, "Relations among Conceptual Knowledge, Procedural Knowledge, and Procedural Flexibility in Two Samples Differing in Prior Knowledge," *Developmental Psychology* 47, no. 6 (2011): 1525–1538. Bethany Rittle-Johnson and Jon R. Star, comparison-of-methods papers, 2007 and 2009: comparison improves flexibility.

comes the answer."

A function is a rule that takes each allowed input to exactly one output. $f(x)$ is the output of f at input x, not f times x. You will meet three main families. Linear: add the same amount each time. Exponential: multiply by the same amount each time. Quadratic: the second differences are constant, and the graph is a parabola. Your job is to tell them apart, and to move among a formula, a table, a graph, and a sentence.

Tiny worked example. Solve $-2x + 5 = 11$.

First, test. If x is -3, the left side is $-2(-3) + 5 = 6 + 5 = 11$. Both sides 11. True. If x is 2, the left side is $-4 + 5 = 1$. $1 = 11$ is false. So -3 works and 2 does not.

Now compress. $-2x + 5 = 11$. Add the opposite of 5 to both sides: $-2x = 6$. Divide both sides by -2: $x = -3$. Substitute back: $-2(-3) + 5 = 11$. True. The inverse steps are a shorter version of the search you already did.

Try 1. Expand $(x + 2)(x + 3)$ two ways. First draw a rectangle with sides $x + 2$ and $x + 3$ and write the four areas. Then distribute: the $x + 2$ hits both terms of $x + 3$. You should get $x^2 + 5x + 6$ both times. If you get $x^2 + 5$, the 2 never multiplied the 3, and the x terms never met.

Try 2. Look at this table: $0 \to 3$, $1 \to 6$, $2 \to 12$, $3 \to 24$. Are the first differences constant, or the ratios? Name the family. Write a formula.

Explain it back. In one or two sentences, tell someone what "equals" means in $-2x + 5 = 11$, and how you would check a proposed x without doing the inverse steps first.

Challenge. $x^2 + 6x + 5$. Rewrite it by completing the square. You need a 9 to make $x^2 + 6x$ a square. 5 is 9 minus 4, so the rewrite is $(x + 3)^2 - 4$. Check by expanding. Then say what the vertex of $y = x^2 + 6x + 5$ would be, and why.

One more picture, same idea. A table $0 \to 0$, $1 \to 1$, $2 \to 4$, $3 \to 9$ is quadratic: first differences 1, 3, 5, second differences constant 2. A table $0 \to 3$, $1 \to 5$, $2 \to 7$, $3 \to 9$ is linear: first differences constant 2. You should be able to name the family before you write the formula. If you write $y = mx + b$ for both, the families have collapsed.

You are allowed to struggle. A wrong expansion is information, not a verdict. Check by substituting. Draw the rectangle when the product goes missing. Name the family before you write the formula. The tool does not take this test. You do.

If it isn't clicking

Three diagnostics, and the next move for each. No shame. Placement is by skill, not by birthday.

Diagnostic 1 — the gates are still open. The child writes $1/2 + (-1/3) = 1/5$, or cannot place $-3/4$, or treats $3 + -2$ as a slogan, or writes $3 + 4 + 5 = 3 + \square$ as 15.

Next move. Pause the Algebra I book. Return to signed fractions on a number line, integer operations as moves, and the equal sign as a relation inside short equations. The Panel's pace judgments put proficient operations with positive and negative integers by the end of Grade 6 and with positive and negative fractions by the end of Grade 7 — judgments from curriculum comparison, not experimental placements.[159] A child who is calendar-grade 9 with those operations still fragile is not "behind a legal grade." The child is at the door. Stay there until substitution and signed rationals are accurate untimed. Then this course.

Diagnostic 2 — a procedure without a picture. The child can isolate x and cannot substitute back. Can FOIL friendly binomials and writes $(x + 2)(x + 3) = x^2 + 5$ on a bad day. Can chant $y = mx + b$ and cannot say how that growth differs from $y = 2$^x.

Next move. Slow down on the new family or the new rewriting. One conceptual pass: the properties still hold, or the rate is additive versus multiplicative. Then a handful of items. Then a return to the property. Put the rectangle next to the product. Put the table next to the graph next to the formula. Compare two methods for the same quadratic once one method is executable. If the child is already lost inside a first two-step equation, start with a fully worked example, not with "see if you can figure it out." Explicit instruction — clear models, practice, think-alouds, feedback — has consistently positive effects for students with mathematical difficulties, and that does not mean every minute of the year must be a lecture.[160] Generation-then-instruction can help on a new concept when priors exist.[161] Missing facts are not a chance to discover. They are a fluency problem.

Diagnostic 3 — choosing is the hole, not executing. The child can factor when

[159] NMAP, *Foundations for Success* (2008), Table 2.

[160] NMAP, *Foundations for Success* (2008), Finding 27. Fuchs et al., *Assisting Students Struggling with Mathematics* (2021), Recommendation 1, systematic instruction.

[161] Manu Kapur, "Productive Failure in Learning Math," *Cognitive Science* 38, no. 5 (2014): 1008–1022. A different comparison from Finding 27: generation-then-instruction on a new concept when students have usable priors.

the page says “factor,” complete the square when the page says “complete the square,” and freezes on a mixed page.

Next move. Keep going on new content in short blocked bursts, and change the practice. Mix yesterday’s type with last month’s. Ask “which family?” before “which formula?” An unannounced mixed quiz once a month. Pull items from earlier chapters of the book onto this week’s page. Choosing is a skill. Blocked pages will not teach it.

When to get a human tutor. You are the ceiling on tonight’s idea, and preparing the five-minute warm-up still leaves you unable to hear the wrong inverse. Or the child has the gates and weeks of faded examples and mixed practice, and the unaided ticket is still a blank. A tutor, a co-op teacher, or a dual-enrollment College Algebra later is a normal high-school plan. In the 2016 survey, 23 percent of homeschoolers had any instruction by a tutor or private teacher; 31 percent had any from a local homeschool group or co-op.[162] Stay the supervisor who can still hear a wrong expansion on the exit ticket, even if someone else taught Tuesday’s lesson. Video programs exist because many adults feel rusty. A program that “does 100 percent of the teaching” is a description of a product, not a finding that you can stop listening.[163]

When to go ahead. Signed rationals are accurate. The equal sign is a relation the child can check by substituting. Structure in an expression is visible — the 3 hits both terms. One family is fluent and a second is in progress. The unaided ticket is mostly right, including on mixed items. Finding 30 of the Panel still allows prepared, motivated students to move faster, with no harm to their learning.[164] Compacted Grade 7–8 so that Algebra I finishes in Grade 8 is a pathway choice, not a kitchen experiment. Diagnose. Then move.

Tools, including AI

Optional helpers for you. The child attempts first. You hold the answer key.

AI box — Algebra I

[162] Cui and Hanson, NCES 2020-001: any instruction by a tutor or private teacher, 23 percent; any from a local homeschool group or co-op, 31 percent.

[163] Teaching Textbooks homepage, fetched 30 August 2026: the program “does 100% of the teaching” and “100% of the grading.” Product description, not an independent effectiveness trial.

[164] NMAP, *Foundations for Success* (2008), Finding 30.

> Use a tool to explain today's idea to *you* (why $(x + 2)(x + 3)$ is not $x^2 + 5$; why $f(x - 2)$ shifts right), to make extra practice with answers hidden from the child, to write a hint script that withholds the finished polynomial, or to diagnose work already done.
>
> The child does tonight's set unaided. No pasting the worksheet, no "what's the answer," no photo-to-key, no unsupervised chatbot during the attempt.
>
> A computer-algebra site is a check after the child can execute by hand, the way a calculator is a check after fluency. It is not the lesson.
>
> Unguarded solvers raise practice scores and then cut unaided exams. A tutor-style tool that withholds the full solution does not show that harm.[165] The exit ticket stays unaided either way.

Adaptive practice engines can sequence items. They do not replace you hearing a wrong expansion. Facts and fluent algebra moves are not a chat job. Use a tool to generate a short oral script. Then you and the child, out loud, no screen between the fact and the mouth.

What "done enough" looks like

A "grade 9 book" is a publisher's scope, not a legal grade. Done enough is skill.

Checklist before moving on.

- Operates on signed rationals, including as coefficients, without turning the page into soup.
- Treats "=" as the same amount, and checks by substituting, including "never true" and "always true."
- Sees structure: the 3 hits both terms; $(x + 2)(x + 3)$ is a product, not a chant.
- Moves among formula, table, graph, and a sentence for linear, exponential, and quadratic functions. Distinguishes the families by rate of change, by

[165] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. GPT Base: assisted practice +48 percent relative to control; unaided exam −17 percent. GPT Tutor (withheld the full solution): practice +127 percent; exam statistically indistinguishable from control. Classroom-level RCT, large high school in Turkey, grades 9–11. Not a homeschool study.

ratios, or by second differences.
- Solves linear equations and systems; graphs inequalities as a set, not only as a single number.
- Factors, completes the square, and uses the quadratic formula with real solutions, and can say which method is friendlier for this problem.
- Recognizes that a negative discriminant means no real solutions, without developing complex numbers yet.
- Graphs an absolute-value V, and can read a simple piecewise or step graph.
- Plots univariate data; reads a two-way table; fits a line informally; says out loud that correlation is not causation.
- On an unaided mixed page, chooses a strategy, not only the one in the worksheet title.

What can still be thin: fancy word-problem cover stories, every piecewise graph in the book, technology-computed r to many decimals. Those can travel into the next course. The function concept cannot.

Transcript. Write **Algebra I**. If the spine was a publisher name, add a parenthetical a stranger can map: "Algebra I (publisher Algebra 1)." The National Collegiate Athletic Association's math core courses are Algebra I or higher.[166] Named college pages this project opened want a transcript with course titles, grades, and credits, and often three years of mathematics through Algebra II as a floor, not a rule.[167] Integrated-path families transcript Integrated Mathematics I, or Mathematics I, with a parenthetical a stranger can map onto Algebra I plus geometry. "Saxon 8/7," "Epsilon," or "Fun with numbers," with no parenthetical, is not a course title a registrar can read.

If the year never reached a function concept, an exponential family, or a quadratic with real solutions, transcript the work that was done. Finish the catalog, or name the year more honestly. Authenticity is content, not a cover.

A parent who felt rusty should be able to say, after this chapter: I know what the equal sign means in a multi-step equation. I know what $(x + 2)(x + 3) = x^2 + 5$ usually means. I know the three questions at the door. I know what to say

[166] NCAA Eligibility Center, Home School Toolkit 2025–26: math core courses are Algebra I or higher. NCAA is a private association's eligibility rules, not a statute.

[167] University of Pittsburgh home-schooled-applicant page: minimum three years of mathematics. Southern Methodist University application guide for home-schooled and nontraditional students: suggested Algebra I, Geometry, Algebra II. North Central College: three years including algebra and geometry, one course beyond geometry preferred. Pattern, not a rule.

tomorrow. The child still does the struggle. We can do this.

Chapter 5

Geometry

Figure 6: A compass, a straightedge, and a paper with a circle and a triangle.

Why this matters

Geometry is the year a child learns to give a reason a stranger could check. Congruence becomes a motion you can describe, not a feeling that two triangles "look the same." Similarity becomes a scale factor on corresponding sides, not a vibe about shape. A drawing that looks right is still a drawing. An argument that names the given parts, the correspondence, and the definition is a proof — in a paragraph, in a flow diagram, in two columns, or in a marked diagram that actually argues.

That is worth the struggle because later mathematics needs it. Algebra II will put sine and cosine on the unit circle; the seed of those functions is the similar right triangles in this course. Calculus will make precise the informal-limit arguments for area and volume that this year plants. Coordinate geometry is Algebra I sitting on the plane: distance, midpoint, parallel and perpendicular slopes, the circle's equation. Skip this course, or reduce it to two-column theater, and those later objects arrive as symbol soup — or they never arrive, because the transcript never showed Geometry.

The course is not two-column theater. Achieve's model for Geometry says the purpose is to formalize and extend geometric experiences from the middle grades, moving toward formal mathematical arguments, with transformations emphasized early.[168] That is a different opening from many older high-school courses that began with axioms and two-column proof of triangle congruence. If you learned SAS as a postulate and spent a semester on two columns, you are not wrong about the mathematics you learned. You are looking at a different instructional opening. This chapter uses the transformation opening because it is the most fully specified public map, and because congruence is a motion you can see before it is a list of letter-triples you can chant. Both openings can produce viable arguments. Both can fail.

Typical placement is Grade 10 after Algebra I, or Grade 9 after Grade 8 Algebra I. In an integrated pathway the same clusters are distributed across Mathematics

[168] Achieve / Common Core writing team, *Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards*, Oregon Department of Education copy, https://www.oregon.gov/ode/educator-resources/standards/mathematics/Documents/math-appendix-a-model-course-pathways.pdf, accessed 30 August 2026. Geometry: "formalize and extend students' geometric experiences from the middle grades… moving towards formal mathematical arguments." Transformations emphasized early. Pathways and courses are models, not mandates.

I–III rather than concentrated in one year.[169] Grade labels are sequence anchors, not prison bars. In the 2016 federal parent survey, geometry had ever been taught to 52 percent of high-school-level homeschoolers; taught that year, 26 percent of the 9–12 band.[170] Algebra I is common. Geometry is not universal. A kitchen that stops after Algebra I because "the state doesn't require it" has a college problem on named university pages that ask for three years of mathematics, and a mathematics problem in every later course that needs a similar triangle or a coordinate proof.

What this course is, in one page: congruence, proof, and constructions — rigid motions; triangle congruence ASA, SAS, SSS from the definition of congruence in terms of those motions; theorems about lines, angles, triangles, parallelograms; compass-and-straightedge constructions. Similarity, proof, and trigonometry — dilations; AA similarity; trigonometric ratios for acute angles; the Pythagorean theorem via similarity. Informal three-dimensional arguments for circumference, area, and volume. Coordinate geometry. Circles with and without coordinates, including radian as a unit of arc length. Applications of probability: independence, conditional probability, two-way tables.[171] Two-column proof of triangle congruence is one format inside one cluster of one critical area. It is not the course.

What this course is not: unit-circle trigonometry (Algebra II); Cavalieri made into a precise limit (calculus); AP Statistics. Right-triangle trigonometry here is similarity. The unit circle comes next year.

Before you start, diagnose. A child who cannot move a triangle by a rotation, a reflection, or a translation, who treats similar as "looks like," who cannot write a ratio of corresponding sides, is not ready, regardless of calendar grade. Proportional reasoning from the middle grades is load-bearing. Algebra I linear equations and quadratics are load-bearing for coordinates and for completing the square when a circle's equation arrives. Then this course. Transcript it as **Geometry**.

[169] Appendix A (n. 168). Traditional pathway: Algebra I, Geometry, Algebra II. Integrated: Mathematics I–III. Compacted versions complete Grade 7, Grade 8, and the first high-school course in Grades 7–8.

[170] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey*, NCES 2020-001 (Washington, DC: NCES, 2019), Tables 8–9. Geometry ever taught during home instruction, 52 percent of high-school-level homeschoolers; taught that year to 26 percent of 9–12, 19 percent of 6–8, 15 percent of 3–5. The word is doing different work at different ages. Algebra I ever-taught at high-school level, 79 percent.

[171] Appendix A (n. 168), Geometry six critical areas.

For the parent: understand it yourself

You do not need to remember a semester of two-column. You do need to understand today's idea well enough to hear "they look the same" offered as SAS, and a numbered paragraph that assumes the conclusion, as validity failures — not as style preferences.

Everyday picture. Two paper triangles. If you can slide, turn, or flip one so it lies exactly on the other, they are congruent. If you can do that after a uniform shrink or stretch, they are similar. The motion is the meaning. The letter-triples (SAS, ASA, SSS, AA) are compressions of motions that work. SSA is not on the list because a motion is not determined: two non-congruent triangles can share that configuration.

Precise picture. Two figures are congruent if one can be taken onto the other by a sequence of rotations, reflections, and translations — the rigid motions that preserve distance and angle.[172] "They look the same" is not that sentence. $\triangle ABC \cong \triangle DEF$ names a correspondence: $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$. Writing $\triangle ABC \cong \triangle FED$ is a different claim. A fluent paragraph that proves the wrong correspondence has not proved the claim on the page.

The familiar criteria follow from that definition. SAS, ASA, and SSS each name enough given parts that a rigid motion exists. HL for right triangles is SSS in disguise: hypotenuse and leg determine the third side by the Pythagorean theorem. AAA is similarity, not congruence — the same shape, not the same size. SSA is the configuration that does not determine a motion.

Similarity is a dilation plus a rigid motion. A dilation is a scaling about a point that multiplies all distances by the same positive constant, the scale factor.[173] If $\triangle ABC \sim \triangle DEF$ with scale factor k, then $DE/AB = EF/BC = FD/CA = k$. AA similarity is the compression: two pairs of corresponding congruent angles determine a similarity of triangles. Right-triangle trigonometry in this course *is* similarity. The trigonometric ratios for an acute angle are the constant ratios of corresponding sides in the family of similar right triangles that share that acute angle.[174] SOH-CAH-TOA is a memory aid for those three ratios, not a third course. Sine of an

[172] Appendix A; Common Core G.CO: congruence defined via rigid motions; ASA, SAS, SSS from that definition.

[173] Appendix A, similarity via dilations; AA similarity.

[174] Appendix A, trigonometric ratios for acute angles as part of the similarity critical area.

acute angle is opposite over hypotenuse. Cosine is adjacent over hypotenuse. Tangent is opposite over adjacent. They do not yet live on the unit circle.

The Pythagorean theorem via similarity: drop an altitude to the hypotenuse of a right triangle and you have three similar right triangles. The altitude creates $a^2 = c \cdot p$ and $b^2 = c \cdot q$ with $p + q = c$, hence $a^2 + b^2 = c^2$.[175] 3-4-5 and 5-12-13 are useful retrieve-able triples. They are not the theorem.

Proof is a viable argument, not a format. The Common Core's Practice 3: construct viable arguments and critique the reasoning of others.[176] Appendix A's instructional note on the line-and-angle theorems encourages multiple ways of writing proofs — narrative paragraphs, flow diagrams, two-column format, and diagrams without words — and asks students to focus on the validity of the underlying reasoning while exploring a variety of formats.[177] Four formats, one criterion. A valid paragraph is a proof. An invalid two-column is not. A diagram that marks a correspondence and a sequence of rigid motions, with the given parts named, can be a proof. A diagram that "looks right" is a drawing.

A ninth-grader who writes, in prose, "$\triangle ABC$ and $\triangle DEF$ have $AB = DE$, $AC = DF$, and included $\angle A = \angle D$. A rigid motion takes A to D and ray AB onto ray DE. Because the included angles match, ray AC lands on ray DF. Because the sides match, C lands on F. So the triangles coincide, hence they are congruent," has produced a viable argument. A ninth-grader who fills two columns with "Given / SAS / Therefore congruent" without a correspondence has produced a format.

Coordinate geometry is Algebra I on the plane. Distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$, the Pythagorean theorem on a coordinate right triangle. Midpoint is the average of the coordinates. Parallel lines have equal slope; perpendicular lines have slopes whose product is -1, for non-vertical lines. The equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$, is the distance formula set equal to

[175] Pythagorean theorem via similarity: altitude to the hypotenuse produces three similar right triangles. Grade 8 already used the theorem and its converse; this course asks why.

[176] Common Core State Standards for Mathematics, Standards for Mathematical Practice 3: construct viable arguments and critique the reasoning of others. NCTM, *Principles and Standards for School Mathematics* (2000): Geometry is "a natural area of mathematics for the development of students' reasoning and justification skills," beginning well before high school. National Research Council, *Adding It Up* (2001), adaptive-reasoning strand.

[177] Appendix A, instructional note on G.CO.9–11: "Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning."

a constant. Completing the square turns $x^2 + y^2 + 4x - 6y - 3 = 0$ into $(x + 2)^2 + (y - 3)^2 = 16$, a circle of radius 4 centered at $(-2, 3)$. That is the same completing the square Algebra I used for a quadratic. The representation changed. The algebra did not.

Circles, without coordinates: an inscribed angle measures half the intercepted arc; an angle inscribed in a semicircle is a right angle; a tangent is perpendicular to the radius at the point of contact. Arc length is a fraction of the circumference. One radian is the central angle that intercepts an arc equal in length to the radius. Algebra II will need that unit on the unit circle. This course introduces it as a circle measurement.

Informal three-dimensional arguments: dissect a circle into sectors and rearrange toward a parallelogram of base πr and height r, so $A = \pi r^2$ — an informal limit, because the more sectors, the closer the rearrangement is to a rectangle. Volume of a pyramid or cone: three pyramids of equal height and equal base-area fill a prism, hence $V = (1/3)Bh$. Cavalieri, informally: solids with the same cross-sectional area at every height have the same volume. A stack of cards shifted sideways has the same volume as the unshifted stack. That is the argument; it is not yet a precise integral. Grade 8 already had the formulas for cylinder, cone, and sphere. This course asks why.

Cross-sections and solids of revolution are pictures calculus will later integrate. Slice a cube parallel to a face, get a square; slice more freely through a vertex, get a triangle. Revolve a rectangle about one side, get a cylinder; revolve a right triangle about a leg, get a cone. Hear a wrong answer without a two-column: “What two-dimensional figure do you get if you slice this right cylinder with a plane not parallel to the base and not through the axis?” An ellipse, typically, if the plane cuts all generators. A child who always says “circle” has a slogan from one slice.

Laws of Sines and Cosines are extension standards on the public map, often living in a richer Geometry or in a fourth course. The Law of Sines is a ratio of a side to the sine of the opposite angle, constant in a given triangle. The Law of Cosines is the Pythagorean theorem with a correction term for the included angle: $c^2 = a^2 + b^2 - 2ab \cos C$. When C is 90°, cosine is zero, and the Pythagorean theorem returns. That last sentence is why the similarity proof of the Pythagorean theorem still has to be taught even if the Law of Cosines is coming. Their absence from a given book is not a defect without checking whether the family’s next course will carry them. Their presence is not automatic proof that the book is “honors.”

Wrong answers you should be able to hear

1. Two-column as the only proof. The child — or the co-op, or last year's memory — marks a valid paragraph wrong for being a paragraph, and marks an invalid two-column right for having two columns.

What it usually means. Format has been upgraded to mathematics. Practice 3 is validity. Two-column is licensed. It is not required. It is not the definition of geometry.

What to say. "I am grading whether the reasons work, not whether the page looks like the book I remember. Name the correspondence. Name the given parts. Do not assume the conclusion in line two. A short paragraph that does that is a proof. A two-column that skips the correspondence is not."

2. "They look the same" offered as SAS (or as congruence at all). Vertices unmarked. No motion named.

What it usually means. Congruence is still a feeling about a drawing, not a rigid motion with a correspondence.

What to say. "Which vertex goes to which vertex? Describe a motion that would take $\triangle ABC$ onto $\triangle DEF$. SAS is a compression of a motion that works. Looking the same is not."

3. SSA cited as a reason. Or AAA offered as congruence.

What it usually means. The child has treated a configuration that does not determine a rigid motion as if it did — or has mixed similarity with congruence.

What to say. "Those three pieces do not determine a rigid motion. Two different triangles can share SSA. AAA is the same shape, not the same size — similarity, not congruence."

4. A drawing used as a reason. "As shown." The altitude that looks like a median. The tangent that looks perpendicular.

What it usually means. The diagram introduced the configuration and was then asked to do the proving. Pies introduced part-whole in fractions and did not by themselves teach that 3/4 is a number. Diagrams introduce configuration and do not by themselves teach that the configuration is necessary.

What to say. "A drawing is not a theorem. Mark the given. Write the reason.

Check on a second figure that looks different and has the same structure."

5. SOH-CAH-TOA as a chant with no ratio of corresponding sides. Similarity skipped because "trigonometry is Algebra II." Cross-multiply on the first two numbers visible.

What it usually means. Right-triangle trigonometry has been detached from similarity. Proportional reasoning is still a trick.

What to say. "Sin θ is the scale factor from the hypotenuse to the opposite side, in every right triangle with acute angle θ. Mark the corresponding sides first. Then write the ratio."

Five-minute parent warm-up

Before the lesson, on paper:

1. Mark two triangles with SAS given. Write a three-sentence paragraph naming the correspondence and the motion. Then write the same argument in two columns. Both should be valid. Neither is "more proof."
2. Write one invalid argument on purpose: cite SSA, or assume a side equal that was never given. Name, in one sentence, why it fails.
3. For a 30°-60°-90° triangle, say why sin 30° = 1/2 without a calculator: every right triangle with a 30° angle is similar to every other.

If you can judge those three, you can hear a wrong correspondence. The child still writes the argument.

How to teach it this week

Run the door.

Proportional reasoning. Can the child see a ratio as a relationship, find a constant of proportionality, and mark corresponding sides? If not, missing-length problems on similar triangles will become cross-multiply on whatever numbers sit nearby. Stay with ratio and scale until k is visible in a table and a graph through the origin.

Rigid motions and informal similarity. Can the child move a triangle by a rotation, a reflection, or a translation, and say what is preserved? Treat similar as a scale factor, not as "looks like"? Grade 8 already used those as the meaning of congruence and similarity. This course uses that meaning as a definition, then derives the familiar criteria.

Algebra I for coordinates. Can the child write the equation of a line, compute a slope from two points, and complete the square? If not, the circle's equation will be soup. Pause for that algebra. Geometry is not the non-algebra year.

When the gates are open, the sitting is the same shape as Algebra I, scaled to an argument.

Warm-up (3–5 minutes, unaided). One rigid-motion image: "Does this sequence take $\triangle ABC$ onto $\triangle DEF$?" One missing-length on similar triangles. One "is this a proof or a drawing?"

Short model (8–10 minutes). A first formal proof is high-element-interactivity material. Show one fully worked argument that a given SAS configuration determines a rigid motion — paragraph or two-column, either licensed — then a second the child must explain, then a completion problem missing the last reason, then a conventional problem, then an incorrect example: a two-column that cites SSA, or a paragraph that assumes the triangles congruent in line two in order to prove them congruent in line six.[178] Applying incorrect examples to a first geometric argument is the same teaching move used in Algebra I on two-step equations, now on a proof. Label it as a transfer of a method, not as a geometry-proof experiment.

Exact wording you can steal.

On congruence: "Congruence is a rigid motion. Which motion would take A to D? Which vertices correspond?"

On a first proof: "Write the definition. Then use it. 'They look the same' is not a definition."

On format: "You may write a paragraph, a flow, two columns, or a diagram that argues. I am reading whether the reasons work."

[178] Julie L. Booth, Karin E. Lange, Kenneth R. Koedinger, and Kristie J. Newton, "Using Example Problems to Improve Student Learning in Algebra," *Learning and Instruction* 25 (2013): 24–34. Incorrect examples especially useful for conceptual understanding, combined with guided practice in an Algebra I Cognitive Tutor. Applying the sequence (full example, child explains, completion, conventional, incorrect example) to a first geometric argument is a transfer of that method. John Sweller and Graham A. Cooper, "The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra," *Cognition and Instruction* 2, no. 1 (1985): 59–89. Alexander Renkl and Robert K. Atkinson, guidance fading.

On similarity: "Mark A to D, B to E, C to F. The scale factor is this side over the corresponding side. Cross-multiply is a consequence of equal ratios, not a belief."

On trigonometry: "Sin θ is opposite over hypotenuse because every right triangle with acute angle θ is similar to every other. The mnemonic is a nickname for that fact."

On a drawing: "What did they use as a reason — a given, a definition, a previous theorem, or the picture?"

On coordinates: "The distance formula does not care how badly the figure was drawn. Compute. Then say what the number means."

On a circle: "The tangent looks perpendicular. Why *must* it be, from the definition of tangent as a line that meets the circle at exactly one point?"

On volume: "Show me three pyramids in the prism. The formula $V = (1/3)Bh$ is a compression of that filling, not a decoration."

First problem for the student. Two triangles with $AB = DE$, $AC = DF$, and included $\angle A = \angle D$. Mark the correspondence. Write a short paragraph proving congruence from a rigid motion. Then, if the book wants two columns, translate the same argument. The child writes. You wait.

One good question. "Which vertices correspond, and how do you know the included angle is included?"

Student attempt, then mixed practice. Two or three arguments or missing-length items of the new type. Then mix: yesterday's SAS with last month's inscribed-angle and last week's distance formula, or "prove congruent" versus "find the missing length" versus "write the circle's equation" versus "read $P(A \mid B)$ off a table." Blocked practice to learn execution. Interleaved practice to choose.[179] The clean classroom trials of interleaving were grade 7. The kitchen rule is still mix after a short blocked introduction.

Exit ticket (5–10 minutes, unaided). A short argument, or a second isomorphic figure, without a model and without a solution manual. One rigid-motion image, one missing-length, one "proof or drawing?" You grade against a key you hold.

[179] Doug Rohrer, Robert F. Dedrick, and Kaleena Burgess, *Psychonomic Bulletin & Review* 21 (2014): 1323–1330. Grade-7 classroom interleaving. The practice of mixing types is used here; the coefficient stays with that trial.

How to fade help. Full argument → child explains a second → last reason blank → conventional → incorrect example named → unaided generation. When the child marks the correspondence without being told, stop pointing at A and D. Bring the diagram back when a circle theorem needs an auxiliary line.

When to stop talking. After the model and one question. A first proof tempts a lecture. The child needs to write. Ask, wait, hint ("Which vertices correspond?"), then, if needed, the first line of the motion, not the finished two-column.

Constructions belong with congruence, not as a drawing unit off to the side. Copying a segment is a translation of length. Copying an angle is a rigid motion of a ray. The perpendicular bisector is the set of points equidistant from two endpoints — a reflection's fixed set. The angle bisector is a ray a reflection across which swaps the sides of the angle. Build the perpendicular bisector. Then prove that every point on it is equidistant from the endpoints. Then use that theorem in a later argument. Iterate. A page of constructions with no argument attached is drawing. A theorem with no figure is a format.

Theorems about lines, angles, triangles, and parallelograms are the rest of the first critical area: vertical angles; alternate interior angles with a transversal across parallels; the triangle angle-sum; the exterior-angle theorem; base angles of an isosceles triangle; opposite sides of a parallelogram. Grade 8 already asked for informal arguments for several of these. This course asks for arguments a stranger could check. "Informal" in Grade 8 meant a diagram, a few sentences, a transformation. "Formal" here does not mean two-column. It means the reasons are named, the correspondence is named, and the conclusion is not assumed in the second line.

On a first circle theorem, put three representations in one sitting: the definition (set of points at distance r from a center), the equation, the graph. Then ask the tangent-radius claim on a coordinate circle, so that perpendicular slopes have something to do. An inscribed angle that looks like half the arc is still a claim. Write why, from the central angle that intercepts the same arc.

On Cavalieri week, put a stack of cards next to the sentence "same cross-section at every height, same volume," in the same visual field. A net of a cylinder next to $S = 2\pi r^2 + 2\pi rh$. Fade when the child is accurate without the net. Bring it back when a sphere needs a picture.

If your own course was axiom-first, name the difference once: this book spends early weeks on rigid motions, then derives SAS, ASA, SSS. An older reprint will list postulates, then prove the rest. You are not behind. Teach the validity criterion in either spine. Add a paragraph proof if the book never writes one. Add a motion if the book never names one.

Practice that actually builds learning

A first congruence criterion needs a short blocked set: several SAS configurations that differ in appearance, so execution can form. Then mix.

Blocked, for a new move. The week AA similarity arrives, several missing-length problems that all want corresponding sides of similar triangles, with the correspondence already marked. The week the distance formula arrives, several compute-the-distance items, with the right triangle sketched on the grid.

Mixed, for when to use it. Later the same week: one SAS, one AA plus a missing length, one "write the circle's equation," one inscribed-angle measure, one "is this a proof?" A page titled "prove congruent" teaches execution of a format. A mixed page teaches choosing a strategy.

Brief retrieval. Special right triangles — 30-60-90 and 45-45-90 — after the similarity is visible, not instead of it. Scale factors $1 : \sqrt{3} : 2$ and $1 : 1 : \sqrt{2}$. Retrieve 3-4-5 and 5-12-13 as triples, then ask whether a triangle with sides 5, 12, 13 is right, and why the converse needs a check. Radian facts this year: $180°$ is π radians, $90°$ is $\pi/2$, $30°$ is $\pi/6$. They free working memory later. They are not yet the unit-circle functions of Algebra II.

One incorrect example to diagnose. A two-column that cites SSA. A paragraph that uses the claim as a reason for the claim. A missing-length that cross-multiplied the wrong sides. A proof that says "as shown." Ask what they thought those three letters guaranteed, or which correspondence they used. Then an isomorphic figure the child does unaided. A second figure is how you test whether the argument rode on the drawing.

Compare methods. Substitution versus a coordinate calculation for "the diagonals of a parallelogram bisect each other." Similarity proof of the Pythagorean theorem versus a dissection. Paragraph versus two-column for the same SAS. After the child can execute one, put a second next to it and ask which is better for this problem, and why.

Kitchen building, folding, a scale model, a sewing pattern, Minecraft — those can motivate and can provide a second representation of a similarity or a volume the child is already being taught as an argument. They do not replace the written reason, the correspondence, or the informal Cavalieri.[180] A child who can bisect a board and cannot prove that the perpendicular bisector is the set of equidistant points has performed, not learned. Spatial play is not proof. A shed does not transcript as Geometry.

Probability in this course is not a Friday worksheet. Two events A and B are independent if $P(A \text{ and } B) = P(A) \cdot P(B)$. Conditional probability $P(A \mid B)$ is $P(A \text{ and } B) / P(B)$ when $P(B) \neq 0$. Forty students: 18 take French, 15 take Spanish, 7 take both. $P(\text{French} \mid \text{Spanish})$ is 7/15, not 7/40. Mutually exclusive is $P(A \text{ and } B) = 0$, which is not the same as independent. Plant this cluster. The full statistics course waits.

Representations belong in the same sitting as the written argument: the marked diagram, the rigid-motion sketch, the dilation with a scale factor named, the coordinate grid, the net, the two-way table.[181] Put the motion next to the sentence. A GeoGebra file that moves a triangle, with no sentence saying which motion and which corresponding parts, is a movie. Fade the diagram when the child is accurate without it. Bring it back when a sphere's informal-limit argument needs a picture.

For the student

Geometry is the year you give a reason a stranger could check. Two triangles are congruent when a slide, a turn, or a flip — or a sequence of those — takes one exactly onto the other. They are similar when that happens after a uniform shrink or stretch. "They look the same" is a feeling. A correspondence of vertices is a claim.

A proof is a viable argument. It can be a paragraph, a flow diagram, two columns,

[180] National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 26: real-world contexts improve performance on similar real-world items; they do not, in the Panel's small high-quality set, improve computation, simple word problems, or equation solving. Nicholas C. Soderstrom and Robert A. Bjork, "Learning Versus Performance," *Perspectives on Psychological Science* 10, no. 2 (2015).

[181] Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics*, WWC 2021006 (IES, 2021), Recommendation 3: representations that match the mathematics, connected to notation in the same visual field, faded for accuracy. Kira J. Carbonneau, Scott C. Marley, and James P. Selig, 2013 meta-analysis: manipulatives versus abstract-only, overall mean d approximately 0.37, under guidance.

or a diagram that actually argues. Two columns are one way to write. They are not the definition of being right.

Tiny worked example. Given $AB = DE$, $AC = DF$, and included $\angle A = \angle D$. Claim: $\triangle ABC \cong \triangle DEF$.

Correspondence: A to D, B to E, C to F. A rigid motion takes A to D and ray AB onto ray DE. Because the included angles match, ray AC lands on ray DF. Because the sides match, C lands on F. The triangles coincide, so they are congruent. That is SAS as a compression of a motion, not as three magic letters.

Try 1. Two triangles look alike. The given parts are two sides and a non-included angle (SSA). Write one sentence saying whether a rigid motion is determined, and why.

Try 2. A right triangle has an acute angle of 30° and hypotenuse 10. Using similarity, not a calculator, find the side opposite 30°. Say why $\sin 30° = 1/2$ in every such triangle.

Explain it back. What is the difference between a drawing that looks right and a proof? Give one sentence. Then say one format besides two-column that can still be a proof.

Challenge. On a coordinate grid, $A(0, 0)$, $B(4, 0)$, $C(4, 3)$. Show that $\angle B$ is a right angle two ways: slopes whose product is -1, and the converse of the Pythagorean theorem on the three distances. Then write the equation of the circle with diameter AC.

A second small try, on volume. Three pyramids of equal height and equal base-area fill a prism. That is why $V = (1/3)Bh$, not because a formula sheet said so. If you can retrieve the formula and cannot show the filling, you have the compression without the argument. Sketch the prism. Say where the three pyramids go.

Two-column is allowed. It is one way to write a reason a stranger could check. If your book wants two columns, translate the SAS paragraph from the tiny example into two columns. If your book never writes a paragraph, write one anyway, once, so you know the reasons still work without the format.

You are allowed to struggle. Mark the vertices before you write. Check a second figure if the first drawing is doing too much work. The tool does not write this proof. You do.

If it isn't clicking

Diagnostic 1 — the gates are still open. The child still treats 2:3 as "two and three," cannot find *k*, or cannot complete the square. Similar-triangle missing lengths are cross-multiply on the nearest numbers. The circle's equation is soup.

Next move. Pause the geometry book on similarity or on coordinates. Return to ratio as a relationship, and to Algebra I's linear equations and completing the square. SOH-CAH-TOA will be a chant until corresponding sides are marked. The circle's equation will be a blob until $(x - h)^2$ is a distance squared. This is not a legal-grade problem. Stay at the door until those two skills are accurate. Then this course.

Diagnostic 2 — format without validity, or motion without argument. The child can fill two columns and skips the correspondence. Or can slide a triangle in GeoGebra and never writes a reason. Or treats a drawing as a theorem.

Next move. Slow down on the first criterion. Definitions first. One faded worked argument. One incorrect example: SSA, or the conclusion assumed in line two, or "as shown." Then the child writes a short unaided paragraph. Grade the paragraph on validity. If the book is two-column-only, keep two-column as a tool and add a paragraph so a valid paragraph is not marked wrong. If the book never writes a reason, add one. Both holes are filled the same way: definition, faded model, incorrect example, generation.

Diagnostic 3 — choosing is the hole. The child can prove SAS when the page says SAS, find a missing length when the page says similar, and freezes on a mixed page or on a figure with extra marks.

Next move. Keep going on new theorems in short blocked bursts. Change the practice: mix yesterday's SAS with last week's distance formula and a "proof or drawing?" item. Ask which correspondence first, every time. An unannounced mixed quiz once a month. A second isomorphic figure after every new theorem, so the argument cannot ride on one drawing.

When to get a human tutor. You cannot yet hear a wrong SSA, and the five-minute warm-up is not closing the gap. Or the child has the gates and weeks of faded arguments, and the unaided paragraph is still a drawing. A co-op geometry class or a weekly tutor is a normal high-school plan — 31 percent of homeschoolers in 2016 had any local group or co-op; 23 percent had any tutor.[182] Stay the judge of the homework's validity, or hire a judge (the co-op teacher as teacher of record), rather than handing the figure to a chatbot. Video programs exist because many adults feel rusty. Watch enough of the lesson to hear a wrong correspondence. The unaided exit ticket still happens.

When to go ahead. Correspondence is marked without a prompt. A short unaided paragraph or two-column names given parts and does not assume the conclusion. Similarity is a scale factor on corresponding sides. Acute-angle sine, cosine, and tangent are ratios, not buttons. Distance, midpoint, and a circle's equation are accurate. Informal Cavalieri can be shown with a stack of cards and a sentence. Finding 15 still refuses the age lock; Finding 30 still allows prepared, motivated students to move faster.[183] Neither finding licenses a skip of similarity because the child is "good at math."

Tools, including AI

Optional helpers for you. The child writes the proof unaided. You, or a human solution manual, are the judge.

> **AI box — Geometry**
>
> Use a tool to explain to *you* what would count as a viable argument (why SAS determines a motion and SSA does not; a short paragraph, not only two-column), to generate a second figure that shares tonight's structure with a key you hold and check, to write a hint script ("Which vertices correspond?" — not the finished proof), or to diagnose an argument the child already wrote.
>
> The child writes tonight's proof unaided. No pasting the figure with "write the two-column," no photo-to-key, no unsupervised chatbot during the attempt.
>
> A model may describe a diagram or propose an auxiliary line as a question. A fluent numbered paragraph from a chatbot is an untrusted draft.

[182] Cui and Hanson, NCES 2020-001: any local homeschool group or co-op, 31 percent; any tutor or private teacher, 23 percent.

[183] NMAP, *Foundations for Success* (2008), Findings 15 and 30.

> Specialized contest systems that solve Olympiad geometry are not a home product, and ordinary chat models are not those systems.[184]
>
> Unguarded solvers raise practice scores and then cut unaided exams.[185] The exit ticket stays unaided.

A drawing tool you check, used offline to test a conjecture, is a representation. A solver that writes the two-column while the child watches is the practice page without the learning. Detectors are a poor way to police a written argument; they flag human writing at high rates.[186] Skip them. Grade the validity yourself.

What “done enough” looks like

A “grade 10 book” is a publisher’s scope, not a legal grade. Done enough is skill.

Checklist before moving on.

- Congruence is a rigid motion with a named correspondence, not “they look the same.”
- SAS, ASA, SSS are used as compressions of motions that work; SSA is refused; AAA is named as similarity.
- Similarity is a dilation plus a rigid motion; corresponding sides yield a scale factor; AA is used with the correspondence marked.
- Acute-angle sine, cosine, and tangent are constant ratios in a similarity family. Special right triangles are retrieved after that fact, not instead of it.
- A short unaided argument — paragraph, flow, two-column, or a diagram that argues — names given parts and does not assume the conclusion. Format is a tool. Validity is the mathematics.

[184] Yuri Chervonyi and colleagues, AlphaGeometry2, 2025, arXiv 2502.03544: specialized neuro-symbolic system (language model paired with a symbolic engine), 84 percent of IMO 2000–2024 geometry problems (42 of 50). DeepMind-adjacent reporting (InfoQ, February 2025): flagship chat models o1 and Gemini Thinking solved none of that IMO-geometry set. A kitchen-table chatbot is not AlphaGeometry. Treat model-written proofs as untrusted drafts.

[185] Hamsa Bastani et al., “Generative AI without Guardrails Can Harm Learning,” *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. GPT Base: assisted practice +48 percent relative to control; unaided exam −17 percent. GPT Tutor: practice +127 percent; exam statistically indistinguishable from control.

[186] Weixin Liang, Mert Yuksekgonul, Yining Mao, Eric Wu, and James Zou, “GPT Detectors Are Biased against Non-Native English Writers,” *Patterns* 4, no. 7 (2023). Average false-positive rate about 61 percent on human TOEFL essays.

- Distance, midpoint, parallel and perpendicular slopes, and the circle's equation, including completing the square, are accurate.
- Inscribed-angle and tangent-radius claims can be argued, not only seen. Radian is a unit of arc length.
- Informal arguments for area and volume (dissection, Cavalieri, an informal limit) can be shown, not only formula-retrieved.
- A two-way table can yield a conditional probability; independent is not confused with mutually exclusive.
- On an unaided mixed page, the child chooses: prove, compute a length, write an equation, or read a table.

What can still be thin: Laws of Sines and Cosines (often a richer geometry or a fourth course), every construction in the book, polished two-column fluency. Those can travel. Viable arguments and similarity cannot.

Transcript. Write **Geometry**. If a publisher integrated geometry inside Algebra 1, Algebra 2, and a later book, the transcript still needs a course a stranger can map onto Geometry.[187] "Math 10" is not enough. Integrated-path families transcript Integrated Mathematics II, or Mathematics II, with a parenthetical a stranger can map onto Geometry plus remaining algebra. NCAA math core courses are Algebra I or higher; Geometry sits inside that "or higher." Named college pages often list Geometry by name in a suggested three-year sequence through Algebra II.[188]

If the year was two-column theater with no similarity, no coordinates, and no three-dimensional arguments, transcript the work that was done. Finish the catalog, or name the year more honestly.

A parent who learned two-column and not transformations should be able to say, after this chapter: I know congruence is a motion. I know a valid paragraph is a proof. I know what SSA usually means. I know what to say tomorrow. The child still writes the argument. We can do this.

---

[187] Saxon classic editions integrate some geometry inside Algebra 1, Algebra 2, and Advanced Math; a separate Geometry course also exists (publisher packaging, Rainbow Resource catalogue). The transcript still needs a course title a stranger can map onto Geometry. Math-U-See and Art of Problem Solving also name a Geometry title. No homeschool-curriculum randomized trial comparing programs was opened for this project. Choose by fit.

[188] NCAA Eligibility Center, Home School Toolkit 2025–26: math core Algebra I or higher. University of Pittsburgh: minimum three years of mathematics. Southern Methodist University: suggested Algebra I, Geometry, Algebra II. North Central College: three years including algebra and geometry, one course beyond geometry preferred. Pattern, not a rule.

Chapter 6

Algebra II and Precalculus

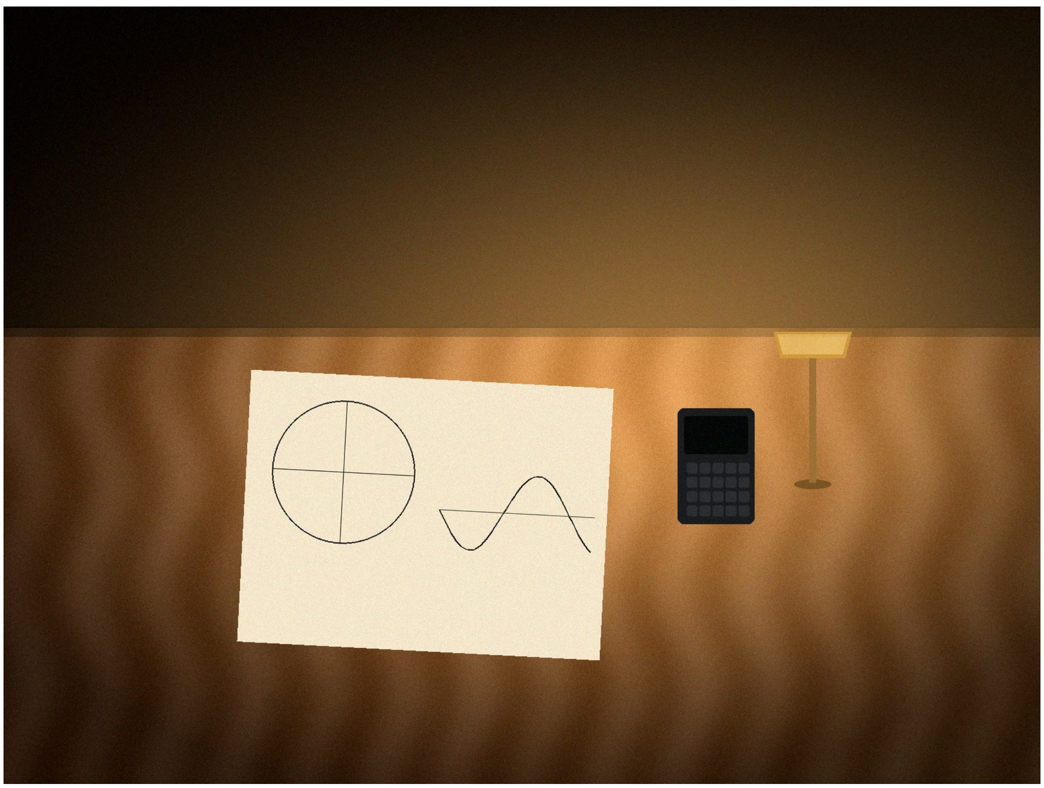

Figure 7: A paper unit circle and sine curve beside a graphing calculator with a blank dark screen.

Why this matters

Algebra II is the year school algebra finishes — or does not. Polynomials go past quadratic. A logarithm stops being a calculator button and becomes the inverse of an exponential. Sine and cosine leave the acute right triangle and become functions of a real number on the unit circle. Complex numbers make it possible to say that a polynomial of degree n has n roots, counting multiplicity. Inverse functions are correspondences run backwards, not the inv key.

That is worth the struggle because calculus is a new operation on functions the student already knows. A student who cannot move among formula, graph, table, and a sentence, who knows trigonometry only as SOH-CAH-TOA, who treats ln and $\log_{10}$ as the same button, will experience $f'(x)$ as symbol soup. A student who can place (cos 2π/3, sin 2π/3), solve ab^{ct} = d as a logarithm, and say that the remainder on division by $x - a$ is $p(a)$ has something for a derivative and a chain rule to attach to.

The course is not a birthday, and precalculus is not a legally defined grade. Achieve's model for Algebra II focuses on rational and polynomial expressions, equations, and functions, plus a continued study of probability and statistics, with more modeling than many older Algebra II courses carried.[189] Typical placement is Grade 11 after Geometry, or Grade 10 after a Grade 8 Algebra I and a Grade 9 Geometry. Those are calendar facts. The National Mathematics Advisory Panel's Finding 15 still refuses the age lock if prerequisites are present.[190] Accelerate a fifteen-year-old who has Algebra I's function fluency and Geometry's right-triangle trigonometry. Wait if those gates are closed, even if the neighborhood is in Algebra II.

At home, Algebra II is not universal. In the 2016 federal parent survey, Algebra II had ever been taught to 49 percent of high-school-level homeschoolers. Calculus

[189] Achieve / Common Core writing team, *Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards*, Oregon Department of Education copy, https://www.oregon.gov/ode/educator-resources/standards/mathematics/Documents/math-appendix-a-model-course-pathways.pdf, accessed 30 August 2026. Algebra II "focuses on rational and polynomial expressions, equations, and functions, as well as a continued study of probability and statistics," with additional modeling emphasis relative to many older Algebra II courses. Pathways and courses are models, not mandates.

[190] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (U.S. Department of Education, 2008), ERIC ED500486, Finding 15.

had ever been taught to 15 percent, a cell that carries a caution flag.[191] Algebra I is common (79 percent). Geometry sits at 52 percent. The drop from Algebra I through this course to calculus is the high-school problem this book is written to name. A family that stops after Algebra I because “the state doesn’t require it” has a college problem on named pages that ask for three years of mathematics through Algebra II. A family that transcripts Algebra II after a year of leftover quadratics and a little log has a mathematics problem the Panel already named.

What this course is: complex numbers; polynomial and rational expressions beyond quadratic, including the remainder theorem; radical and rational equations, with extraneous solutions checked; polynomial, rational, radical, exponential, logarithmic, and trigonometric functions; inverse functions; unit-circle trigonometry and radian measure; logarithms as solutions of exponentials; the normal model and high-school inference from surveys, experiments, and observational studies.[192] That is a year. It is a lot because the course finishes school algebra as the Panel defined it.

What a book titled **Precalculus** is usually buying is a publisher’s own scope, not a statute. The public map’s “fourth course” column holds most of the extra standards: vectors and matrices; further inverse-function work; further trigonometry; remaining conics; polar coordinates and parametrics; further probability.[193] Shopping for precalculus usually means that remainder plus a tightening of Algebra II functions. AP Precalculus is one named fourth-course object, first widely offered 2023–24, with modeling and functions as the published aim and three of four units assessed on the exam.[194] Credit and placement are set by each college. Dual-enrollment

[191] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey*, NCES 2020-001 (Washington, DC: NCES, 2019), Table 8. Algebra II ever taught, 49 percent of high-school-level homeschoolers; calculus, 15 percent (interpret with caution); geometry, 52 percent; basic algebra (Algebra I), 79 percent. The 2023 First Look (NCES 2024-113) did not republish those subject tables.

[192] Appendix A (n. 189), Algebra II cluster homes: N.CN.1–2 and use in polynomial identities and equations; A.SSE; A.APR.2–3 (zeros and factors; remainder theorem), A.APR.6 (rewrite rational expressions); A.REI.2 radical and rational; A.REI.11 graphs combining function types; polynomial, rational, radical, exponential, logarithmic, trigonometric functions; inverse functions; F.TF.1–2 unit circle and radian, F.TF.5 periodic, F.TF.8 Pythagorean identity; F.LE.4 logarithms as solutions of exponentials; S.ID.4 normal; S.IC inference.

[193] Appendix A, “Fourth Courses” column: most extra standards not required for college-and-career readiness as the Appendix defines it — vectors and matrices; remaining complex-plane work; further inverse functions; further trigonometry; remaining conics; further probability and expected value. Precalculus is not a legally defined course.

[194] College Board, AP Precalculus course page, https://apcentral.collegeboard.org/courses/ap-

College Algebra or Precalculus is a third object: a college transcript admissions offices already know how to read.

The Panel's catalog of school algebra, independent of packaging, is the honest checklist: symbols and expressions; linear equations; quadratic equations; functions through trigonometric, including fitting models to data; algebra of polynomials, including complex numbers and the fundamental theorem of algebra; combinatorics and finite probability.[195] Algebra I was the first half. This chapter is the remainder. A homeschool Algebra II that omits functions, logarithms, and trigonometry has not covered school algebra as the Panel defined it. Transcript the work that was done. Write **Algebra II** for the third course, **Precalculus** for a fourth course that earned that name — not "Math 11," not a publisher spine without a parenthetical a stranger can map.

For the parent: understand it yourself

You do not need to remember every identity. You do need to understand today's idea well enough to hear $\log(a + b)$ written as $\log a + \log b$ as a smashed product rule, and $\sin^2\theta + \cos^2\theta$ written as $\sin(2\theta)$ as a smashed identity — not as cute slips.

Everyday picture. A logarithm is an exponent. $\log_{10} 100 = 2$ because $10^2 = 100$. $\ln 100$ is not 2, because e^2 is not 100. The two buttons differ because the bases differ. The unit circle is a circle of radius 1. Walk a distance θ from $(1, 0)$ along the circle. The point you land on is $(\cos\theta, \sin\theta)$. Sine and cosine are the coordinates. That is why they can take any real input, not only an acute angle in a right triangle.

Precise picture. A function f assigns to each input exactly one output. An inverse

precalculus, accessed 30 August 2026. First widely offered 2023–24; 2026–27 clarifications: "Course content has not changed." Four official units with published multiple-choice weights: Unit 1 Polynomial and Rational Functions 30%–40%; Unit 2 Exponential and Logarithmic Functions 25%–40%; Unit 3 Trigonometric and Polar Functions 30%–35%; Unit 4 Functions Involving Parameters, Vectors, and Matrices not assessed. Units 1–3 are on the exam. Exam-format changes take effect May 2027. Credit and placement are set by each college; College Board itself says to search each college's policy.

[195]NMAP, *Foundations for Success* (2008), Table 1, Major Topics of School Algebra: Symbols and Expressions (polynomial and rational expressions; arithmetic and finite geometric series); Linear Equations; Quadratic Equations; Functions (linear, quadratic, polynomial, simple nonlinear, exponential, logarithmic, trigonometric, and fitting models to data); Algebra of Polynomials (including complex numbers and the fundamental theorem of algebra); Combinatorics and Finite Probability. Catalog of algebra through Algebra II, independent of course packaging.

f^{-1}, when it exists, assigns to each output of f the unique input that produced it. $f^{-1}(f(x)) = x$ on the appropriate domain. The superscript −1 is not an exponent. f^{-1} is not $1/f$. Not every function has an inverse that is a function. $f(x) = x^2$ sends both 2 and −2 to 4. Restrict the domain to $x \geq 0$, and the inverse is the principal square root. The graph of f^{-1} is the graph of f reflected across $y = x$. The table of the inverse is the table of f with the columns swapped.

Exponential functions: *f*(*x*) = *b*^*x*, $b > 0$, $b \neq 1$. Domain all reals. Range positive reals. $f(0) = 1$. Growth when $b > 1$; decay when $0 < b < 1$. “Exponential means it grows fast” fails for (1/2)^*x*. Logarithmic functions are the inverses: *y* = log_b *x* means *b*^*y* = *x*. Domain positive reals. Range all reals. log_b 1 = 0; log_b *b* = 1. Change of base: log_b *x* = log_k *x* / log_k *b*.

The three log laws are compressions of exponent rules, because a logarithm is an exponent. Product: log_b(*xy*) = log_b *x* + log_b *y*, same base. Quotient: log_b(*x*/*y*) = log_b *x* − log_b *y*. Power: log_b(*x*^*p*) = *p* log_b *x*. Derive them from *b*^{log_b x} = *x*. They are not a list to be laminated. ln and $\log_{10}$ differ because they are logarithms in different bases: $\log_{10} 100 = 2$, ln 100 = 2 ln 10, and ln 10 is not 1. Calculus will prefer *e* because the derivative of *e*^*x* is *e*^*x*. Solving *ab*^{*ct*} = *d* as a logarithm is the modeling move: 3 · 2^{0.5*t*} = 24 becomes 2^{0.5*t*} = 8, so $0.5t = \log_2 8 = 3$, so $t = 6$ when the numbers cooperate; change of base when they do not.

The unit circle extends trigonometry past Geometry’s acute ratios.[196] One radian is the directed arc length on a circle of radius 1. A full turn is 2π. 180° is π radians; 90° is $\pi/2$; 30° is $\pi/6$; 45° is $\pi/4$; 60° is $\pi/3$. $(\cos\theta, \sin\theta)$ is the point, counterclockwise from the positive *x*-axis. Cosine is the *x*-coordinate. Sine is the *y*-coordinate. Tangent is sine over cosine, when cosine is not zero. SOH-CAH-TOA is the first-quadrant special case, recovered when the terminal ray cuts the unit circle in the first quadrant and you drop a perpendicular: opposite is $\sin\theta$, adjacent is $\cos\theta$, hypotenuse is 1. The mnemonic did not become false. It became a special case.

The Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ is the equation of the unit circle, $x^2 + y^2 = 1$, with $x = \cos\theta$ and $y = \sin\theta$. It is not a new fact. It is the circle. Periodic modeling — $y = A \sin(B(x - C)) + D$ — has amplitude $|A|$, period $2\pi/|B|$, phase

[196] Appendix A, F.TF.1–2, 5, 8 in Algebra II. Remaining trigonometric standards often sit in a fourth course. Geometry already taught trigonometric ratios for acute angles; this course extends them to functions of a real variable.

shift C, midline $y = D$. A tide table is a legitimate model. It is not a substitute for the definition.

Complex numbers: $a + bi$, with $i^2 = -1$. Add real parts and imaginary parts. Multiply using the distributive property and replace i^2 with -1. $(2 + 3i)(1 - i) = 2 - 2i + 3i - 3i^2 = 5 + i$, because $-3i^2 = 3$. $x^2 + 1 = 0$ has solutions i and $-i$. $x^2 - 2x + 2 = 0$ has discriminant -4, so $1 \pm i$. Algebra I recognized that the discriminant can be negative. This course writes the solutions. The fundamental theorem of algebra, in the form a high-school course can use: a polynomial of degree n has n roots in the complex numbers, counting multiplicity. $x^3 - 1 = 0$ has three roots, one real and two complex. A kitchen does not need to prove the theorem. A kitchen can use it as a counting principle.

The remainder theorem: for a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$.[197] Division of $x^3 - 1$ by $x - 1$ produces remainder 0, and $p(1) = 0$, which is why $x - 1$ is a factor. Division by $x - 2$ produces remainder $p(2) = 7$. The remainder *is* the value. Synthetic division is a compression of the same arithmetic. If $p(a) = 0$, then $x - a$ is a factor, and conversely.

Rational expressions are quotients, not decorations to be crossed out. Rewrite $a(x)/b(x)$ as $q(x) + r(x)/b(x)$, the same move as $7 \div 3 = 2$ remainder 1, written $2 + 1/3$, now with polynomials. Cancelling the x in $(x + 2)/(x + 3)$ to get $2/3$ is treating letters as marks to erase. Radical and rational equations can produce extraneous solutions because squaring is not reversible on the whole real line the way adding 3 is. Checking by substituting back is the definition of solving, still. The operational equal sign, which wanted a number on the right, still has nowhere to go.

End behavior of polynomials: odd degree, positive leading coefficient — down on the left, up on the right. Even degree, positive leading coefficient — up on both sides. A rational function $r(x) = 1/(x - 2)$ has a domain that excludes 2. Those are features of graphs the student already learned to read, now on new families.

Wrong answers you should be able to hear

1. Log laws as slogans. $\log(a + b)$ written as $\log a + \log b$. $\ln(ab)$ written as $\ln a \cdot \ln b$. $\log(a + b)$ treated as a product. ln and $\log_{10}$ treated as synonyms. log_b(xy) written with mismatched bases.

[197] Appendix A, A.APR.2 remainder theorem; A.APR.3 zeros and factors.

What it usually means. The child has a laminated list and not the inverse relationship. A logarithm turns a product into a sum *because* it is an exponent, and only when the bases match.

What to say. "A logarithm is an exponent. log_b(xy) = log_b x + log_b y because $b^{\{m+n\} = b}m \cdot b$^n, same base. There is no law for $\log(a + b)$. What is $\log_{10} 100$? What is ln 100 in terms of ln 10? Those are not the same number."

2. $\sin^2\theta + \cos^2\theta$ written as $\sin(2\theta)$, or as $\sin\theta + \cos\theta = 1$. Or $\sin^{-1}(x)$ written as $1/\sin(x)$.

What it usually means. A square of a function has been smashed with a function of a double, or the squares were dropped. The inverse-function superscript has been read as a reciprocal.

What to say. "$\sin^2\theta + \cos^2\theta$ is the unit circle, $x^2 + y^2 = 1$. It is not a double-angle. f^{-1} is the inverse, not $1/f$. Write arcsin if the superscript is picking a fight."

3. ($\cos 2\pi/3$, $\sin 2\pi/3$) placed in quadrant I, or SOH-CAH-TOA treated as the whole of trigonometry.

What it usually means. Sine is still only an acute ratio. The reference angle was used without the sign of the coordinate.

What to say. "Cosine is the x-coordinate. In quadrant II the x-coordinate is negative. Place the point. SOH-CAH-TOA is the first-quadrant special case, not the function."

4. Remainder 0 next to $p(2) = 7$, or "cancel everything that looks alike" in a rational.

What it usually means. Division and substitution have not met. Letters are being cancelled as marks, not as factors of a quotient.

What to say. "The remainder *is* $p(a)$. Write $p(2)$ next to the remainder line and ask why those two numbers must be the same. Cancelling x in $(x + 2)/(x + 3)$ is erasing marks. Factor, then cancel common factors, then say what is still excluded from the domain."

5. f^{-1} treated as the inv key, or every function still $y = mx + b$.

What it usually means. The correspondence is still "the equation," not each input exactly one output, run forwards and backwards.

What to say. “Swap the columns of the table. If two inputs had the same output, the swap is not a function. $f(x+1)$ is not $f(x)+1$. Check on $f(x) = 2$^x.”

Five-minute parent warm-up

Before the lesson, on paper:

1. Write why $\log_{10} 100 = 2$ and why ln 100 is not 2. Derive log_b(xy) = log_b x + log_b y from exponent rules, same base.
2. Place (cos 2π/3, sin 2π/3) without a calculator. Say why cosine is negative. Check $\sin^2 + \cos^2 = 1$.
3. For $p(x) = x^3 - 1$, compute $p(2)$ by substituting. Then divide by $x - 2$ and circle the remainder. They must match.

If those three are fluent for you, you can hear a wrong inverse and a slogan log law. The child still holds the pencil.

How to teach it this week

Run the door. Three questions, content not birthday.

Fractions, including signed, still. Can the child operate on $(2/3)x^3 - (1/2)x + 1/4$? Place −3/4 and 2/3 on the same line? Polynomial long division still needs fraction fluency. The Panel named fractions, including negative fractions, as the most important foundational skill not currently developed among American students.[198] The 2007 Algebra I teacher survey’s top content gap was rational numbers. There is no later Algebra II replication in the sources this book used. The arithmetic did not retire because the cover said Algebra II.

The function concept from Algebra I. Can the student move among formula, graph, table, and verbal? Assign to each input exactly one output, and say when a graph fails that test? Distinguish $f(x+1)$ from $f(x)+1$, now on exponentials and on sine? Without that concept, calculus’s $f'(x)$ has nothing to attach to.

[198] NMAP, *Foundations for Success* (2008), Executive Summary: “knowledge of fractions is the most important foundational skill not currently developed among American students.” 2007 nationally representative survey of 743 Algebra I teachers: poorest incoming preparation in rational numbers. Robert S. Siegler et al., “Early Predictors of High School Mathematics Achievement,” *Psychological Science* 23, no. 7 (2012): 691–697: elementary fraction knowledge uniquely predicted high-school algebra after controls. Prediction finding, not a curriculum experiment.

Geometry's right-triangle trigonometry and similarity. Can the student write a trigonometric ratio for an acute angle as a ratio of corresponding sides, not as a button? Place a reference triangle inside a unit circle and say why cosine is the x-coordinate? If SOH-CAH-TOA is the whole of trigonometry, this year's unit circle is not a review. It is the first time sine is a function of a real variable.

When the gates are open, the sitting is the same shape.

Warm-up (3–5 minutes, unaided). One inverse from a table. One unit-circle value. One "which function family is this graph?"

Short model (8–10 minutes). First pass through a new symbol-heavy procedure — a log law, a remainder-theorem division, a unit-circle evaluation, an inverse-function swap — one fully worked example you already understand, a second the child must explain, a completion missing the last step, a conventional problem, then an incorrect example.[199] Incorrect examples this year: dropped negative signs; "cancel everything that looks alike"; $\log(a + b) = \log a + \log b$; $\sin^{-1}$ written as 1/sin; $\sin^2\theta + \cos^2\theta$ written as $\sin(2\theta)$; ln and $\log_{10}$ treated as synonyms. The child names the error. Then an isomorphic item. Then the child generates.

A child already lost inside the unit circle needs a worked placement of ($\cos 2\pi/3$, $\sin 2\pi/3$), not a week of unguided discovery. A child who has priors — right-triangle trig, the function concept, the circle's equation — can spend twenty minutes trying to extend sine past 90° with what they already know, then be taught. That is generation-then-instruction, designed. It is not leaving them stuck. Explicit instruction still helps students already behind.[200] Those two comparisons are not the same. Use the faded model when the child is a novice in *this* identity.

Exact wording you can steal.

On logs: "A logarithm is an exponent. log_b(xy) turns a product into a sum because that is what adding exponents does. Same base. There is no law for a sum

[199] John Sweller and Graham A. Cooper, "The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra," *Cognition and Instruction* 2, no. 1 (1985): 59–89. Alexander Renkl and Robert K. Atkinson, guidance fading. Julie L. Booth, Karin E. Lange, Kenneth R. Koedinger, and Kristie J. Newton, "Using Example Problems to Improve Student Learning in Algebra," *Learning and Instruction* 25 (2013): 24–34. Sequence stolen as a teaching move; those papers are Algebra I and earlier, not an Algebra II trial.

[200] NMAP, *Foundations for Success* (2008), Finding 27: explicit instruction for students with mathematical difficulties. Manu Kapur, "Productive Failure in Learning Math," *Cognitive Science* 38, no. 5 (2014): 1008–1022: generation-then-instruction on a new concept when priors exist. Different comparisons.

inside the log."

On ln versus $\log_{10}$: "$\log_{10} 100 = 2$ because $10^2 = 100$. $\ln 100 = 2 \ln 10$. $\ln 10$ is not 1. Different bases."

On the unit circle: "Walk arc θ from (1, 0). The point is (cos θ, sin θ). Cosine is *x*. Sine is *y*."

On the identity: "$\sin^2\theta + \cos^2\theta = 1$ is $x^2 + y^2 = 1$. Given $\sin\theta = 3/5$ in quadrant II, cosine is negative. Compute."

On the remainder: "The last number in the division *is* $p(a)$. Substitute to check. If they disagree, the division did not finish."

On inverses: "Swap the columns. If two *x*-values had the same *y*, the swap is not a function. f^{-1} is not $1/f$."

On a family: "Graph this. Polynomial, rational, exponential, logarithmic, trigonometric? End behavior first, then intercepts, then a formula."

First problem for the student. Place (cos 2π/3, sin 2π/3) without a calculator. Write both coordinates. Check $\sin^2 + \cos^2 = 1$. Say why cosine is negative. The child writes. You wait.

One good question. "You used a 60° reference triangle. What does the second quadrant do to the *x*-coordinate, and why?"

Student attempt, then mixed practice. Two or three items of the new type. Then mix: yesterday's remainder theorem with last month's inverse and this week's Pythagorean identity, plus one "which family?" After a short blocked introduction of a new family, mix yesterday's polynomial graphs with last month's exponentials and this week's rationals.[201] The clean classroom evidence for interleaving is grade 7. The kitchen rule is the same. Compare methods after execution exists: change of base versus a same-base rewrite to solve an exponential; reference angle plus sign versus placing the point on the circle; factor versus remainder theorem versus graph to find a zero.

Exit ticket (5 minutes, unaided). A mixed three-item check: one inverse or log, one unit-circle coordinate, one remainder or "which family?" No tool. You grade

[201] Doug Rohrer, Robert F. Dedrick, and Kaleena Burgess, *Psychonomic Bulletin & Review* 21 (2014); Doug Rohrer, Robert F. Dedrick, Marissa K. Hartwig, and Chi-Ngai Cheung, *Journal of Educational Psychology* (2020). Grade-7 classroom interleaving. The practice of mixing types is used here; the coefficients stay with those trials.

it.

How to fade help. Full example → child explains → last step blank → conventional → incorrect slogan named → unaided generation. A unit-circle drawing is a representation, not a toy. Fade when the child can retrieve (cos π/3, sin π/3) without it. Revisit the circle when retrieval fails.

When to stop talking. After one model and one question. Inverse functions, log laws, and the unit circle are the year many kitchens discover that tonight's idea is the ceiling. Prepare the five-minute warm-up. Then the child generates. Hearing a wrong inverse is the job. Taking the pencil is not.

Compound interest and a Ferris wheel are legitimate models. They help similar contexts. They do not, by themselves, produce ln or radians.[202] Name the family first. Then, if you want, tell the story.

Practice that actually builds learning

Blocked, for a new move. The day the product rule for logs arrives, a short set of same-base products, so execution can form. The day special angles on the unit circle arrive, several placements in one quadrant, then the same angles in a second quadrant.

Mixed, for when to use it. Later the same week: graph this — is it polynomial, rational, exponential, logarithmic, trigonometric? Mix a remainder-theorem item, an inverse from a table, a unit-circle coordinate, a Pythagorean-identity recovery, and a "solve *ab^{ct}* = *d* as a logarithm." A page of twenty identical log-law items teaches execution. Choosing a strategy is the interleaved job.

Brief retrieval of facts already known. Special-angle coordinates in all four quadrants, in radians, without a calculator. $i^2 = -1$. log_b 1 = 0 and log_b *b* = 1. Signed-fraction arithmetic that still sits under a cubic. Time only facts the child can already get right untimed. A language model that will happily invent $\sin^2\theta + \cos^2\theta = \sin(2\theta)$ is the wrong object to put between a child and an identity. Your ears, a short mixed oral, or a dumb spaced-repetition deck.

One incorrect example to diagnose. log(*a* + *b*) = log *a* + log *b*. ln and log treated as the same function. (cos 2π/3, sin 2π/3) in quadrant I. Remainder 0 next to *p*(2)

[202] NMAP, *Foundations for Success* (2008), Finding 26. Nicholas C. Soderstrom and Robert A. Bjork, "Learning Versus Performance," *Perspectives on Psychological Science* 10, no. 2 (2015).

= 7. $\sin^{-1}$ as 1/sin. Ask what they thought the law or the superscript meant. Then an isomorphic item unaided.

Compare methods. After the child can execute one method, put a second next to it. Factor versus remainder theorem versus graph to find a zero. Change of base versus rewriting to a common base. Reference-angle chant versus placing the point and reading the coordinates. Flexibility is a third outcome.

Finite geometric series, the sum $1 + r + r^2 + \ldots + r^{n-1} = (1 - r^n)/(1 - r)$ for $r \neq 1$, is a compression this course can finish. Infinite series wait for a second calculus course. The binomial theorem, $(x + y)^n$, is a named extra on the public map. A kitchen that never writes $(x + y)^3$ from Pascal's triangle has a hole to mark, not an automatic failure of the whole year.

Statistics still live here: use mean and standard deviation to fit a normal when the shape warrants it, and say when the procedure is not appropriate; make inferences from sample surveys, experiments, and observational studies, with randomization and simulation as the high-school tools.[203] The empirical rule (about 68 / 95 / 99.7 percent within one, two, three standard deviations) is a compression *when the normal is a reasonable fit*. A bimodal distribution or a heavy skew is a reason to refuse the model. Independent is still not mutually exclusive. Algebra I already said correlation is not causation. This course adds a specific model and a survey / experiment / observational-study split. AP Statistics is a different college-equivalent whose published prerequisite is first-year algebra; it is a legitimate add-on, not a stealth substitute for calculus in a STEM calculus sequence.[204]

Kitchen, money, and a tide app can be second pictures. They do not replace the inverse relationship, the unit-circle definition, or the unaided page. A child who can watch a tide app and cannot place (cos θ, sin θ) has performed, not learned.

[203] Appendix A, S.ID.4 and S.IC.

[204] College Board, *AP Statistics Course and Exam Description*, Effective Fall 2026: equivalent to a one-semester, introductory, non-calculus-based college course; prerequisite is successful completion of a first-year algebra course. Revised framework encourages taking both AP Statistics and AP Calculus when the student has the background; not published as a substitute for calculus in STEM calculus sequences.

For the student

This course is the year a logarithm is an exponent, and sine is a coordinate. A function still takes each allowed input to exactly one output. An inverse, when it exists, runs that correspondence backwards. f^{-1} is not $1/f$.

The unit circle has radius 1. The real number θ is how far you walk from (1, 0). The point you land on is $(\cos\theta, \sin\theta)$. That is why sine and cosine make sense for $2\pi/3$, and for 7, and for $-\pi$, not only for an acute angle in a right triangle. SOH-CAH-TOA was the first-quadrant special case. It still works there. It is not the whole story.

Tiny worked example. Place $(\cos 2\pi/3, \sin 2\pi/3)$.

$2\pi/3$ is 120°, a second-quadrant angle. The reference angle is $\pi/3$, whose cosine is 1/2 and sine is $\sqrt{3}/2$. In quadrant II, x is negative and y is positive. So the point is $(-1/2, \sqrt{3}/2)$. Check: $(-1/2)^2 + (\sqrt{3}/2)^2 = 1/4 + 3/4 = 1$. That check is the circle.

Try 1. Rewrite 3 · 2^{0.5*t*} = 24 as a logarithm and solve. First divide to get 2^{0.5*t*} = 8. Then $0.5t = \log_2 8 = 3$, so $t = 6$. Say in one sentence why this is an inverse, not a button.

Try 2. $\log(a + b)$ is sitting next to $\log a + \log b$ on a page as if they were equal. Write one sentence saying whether they are, and why. Give a numerical counterexample with base 10 if you can.

Explain it back. Why do ln and $\log_{10}$ differ? Give one numerical example. Then say what $\sin^2\theta + \cos^2\theta$ equals, and why that is the unit circle rather than a double-angle formula.

Challenge. $p(x) = x^3 - 1$. Compute $p(2)$ by substituting. Divide $p(x)$ by $x - 2$ and show that the remainder matches $p(2)$. Then say what remainder you would expect on division by $x - 1$, and why $x - 1$ is a factor.

A second small try, on families. Sketch, from memory, *y* = 2^*x*, $y = \log_2 x$, and $y = x^3$. Which goes to 0 on the left? Which is the reflection of which across $y = x$? Which is a polynomial, and how do you know from end behavior? If all three drawings look like “a curve,” the families have not separated yet. Put a table next to each sketch.

You are allowed to struggle. Check a log law by rewriting it as exponents. Check a unit-circle point by squaring and adding. Check a remainder by substituting. The tool does not take this test. You do.

If it isn't clicking

Diagnostic 1 — the gates are still open. Signed-fraction coefficients turn a cubic into soup. The child cannot move among graph, table, formula, and a sentence. SOH-CAH-TOA is the whole of trigonometry, and (cos 2π/3, sin 2π/3) cannot be placed.

Next move. Pause the Algebra II book on the topic that depends on the hole. Return to signed rationals, to Algebra I's function concept (each input exactly one output, three representations), or to Geometry's similar right triangles. Buying a book titled Precalculus will not close those holes. A shaky correspondence becomes six families the child cannot tell apart. Stay at the door until substitution, representations, and acute ratios are accurate. Then this course.

Diagnostic 2 — slogans without the inverse or the circle. The child can recast 2^x = 8 as x = 3 and cannot recast 2^x = 10 as $x = \log_2 10$. Presses LOG and reads 2 for 100, and cannot say why. Writes log(a + b) = log a + log b. Places special angles in quadrant I only. Writes $\sin^{-1}$ as 1/sin.

Next move. Slow down on the inverse relationship and on the unit-circle definition. One conceptual pass: a logarithm is an exponent; cosine is the x-coordinate. Then a handful of items. Then a return to the definition. Incorrect examples named, then isomorphic items. Derive one log law from exponent rules instead of starting with a list. Place one point on the circle by walking, then retrieve the special angles. If the child is a novice in this identity, start with a worked example, not with "see if you can figure it out."

Diagnostic 3 — choosing is the hole. The child can run synthetic division when the page says synthetic division, expand a log when the page says expand, and freezes on a mixed page or on "which family is this graph?"

Next move. Keep going on new families in short blocked bursts. Change the practice. Mix yesterday's remainder theorem with last month's inverse and this week's unit-circle coordinate. Ask "which family?" before "which formula?" An unannounced mixed quiz once a month. Graph polynomial, rational, exponential, logarithmic, and trigonometric on the same week once each family exists.

When to get a human tutor. Tonight's idea is the ceiling after the five-minute warm-up, week after week — inverse functions, log laws, and the unit circle are the usual wall. Or the child has the gates and weeks of faded examples, and the unaided ticket is still a blank. A weekly tutor, a co-op, or dual-enrollment College Algebra or Precalculus is a normal high-school plan. In 2016, 23 percent of homeschoolers had any tutor; 31 percent had any co-op.[205] Community-college dual enrollment produces a transcript NCAA and admissions offices already know how to read; credit is set by each receiving college. Stay the administrator and the ear on the unaided exit ticket, even if someone else explained the unit circle on Tuesday. Video programs exist because many adults feel rusty. A program that "does 100 percent of the teaching" is a product description, not a finding that you can stop listening.[206]

When to go ahead. The remainder *is* $p(a)$. Complex solutions of a quadratic can be written. An inverse can be found from a table, and the child can say when the swap fails. *ab*^{*ct*} = *d* can be solved as a logarithm, and ln versus $\log_{10}$ can be distinguished. $(\cos 2\pi/3, \sin 2\pi/3)$ sits in quadrant II without a calculator. $\sin^2\theta + \cos^2\theta = 1$ is used in a named quadrant. One period of sine can be sketched from the circle. A normal model can be refused when the shape does not warrant it. Then a fourth course, or a first calculus course, is a conversation, not a leap. Finding 30 still allows prepared, motivated students to move faster.[207] Compact only if the gates are open. A non-accelerated student who starts Algebra I in Grade 9 finishes Algebra II in Grade 11 and has senior year for precalculus, statistics, or a first calculus course — not both a full precalculus and a second calculus course unless the year is overloaded.[208] That is a pathway fact, not a mandate to compact.

Tools, including AI

Optional helpers for you. The child attempts first. You hold the answer key.

[205]Cui and Hanson, NCES 2020-001: any tutor or private teacher, 23 percent; any local home-school group or co-op, 31 percent.

[206]Teaching Textbooks homepage, fetched 30 August 2026: "does 100% of the teaching" and "100% of the grading." Product description, not an independent effectiveness trial. No homeschool-curriculum randomized trial comparing Algebra II programs was opened for this project. Choose by fit.

[207]NMAP, *Foundations for Success* (2008), Finding 30.

[208]Appendix A (n. 189): the regular three-year pathway plus a fourth course reaches precalculus as a junior and calculus as a senior only if high-school content started in Grade 8. Model, not a mandate.

AI box — Algebra II and Precalculus

Use a tool to explain to *you* why ln and $\log_{10}$ differ, and why the unit circle is the right home for trigonometry; to make extra practice with answers hidden; to write a hint script that withholds the finished identity; or to diagnose work already done ($\log(a + b)$ as a smashed product rule; remainder and $p(a)$ that have not met).

The child does tonight's set unaided. No pasting the worksheet or the precalculus test, no "what's the answer," no photo-to-key, no unsupervised chatbot during the attempt.

A computer-algebra site is the right *check* for an identity after the child has attempted, the way a calculator is a check after fluency. A language model is the explainer for you. Mixing the two without a label is how a fluent wrong identity survives.

If AP Precalculus is sat, free response is unaided, on paper. Unguarded solvers raise practice scores and then cut unaided exams.[209] The exit ticket stays unaided.

Facts and fluent algebra are not a chat job. Special-angle coordinates, log laws as retrieved compressions, $i^2 = -1$ belong to your ears or a short mixed oral. Use a tool to generate the oral script. Then close the screen.

What "done enough" looks like

A "grade 11 book" is a publisher's scope, not a legal grade. Precalculus is not a legally defined course. Done enough is the Panel's catalog, not the cover.

Have we finished Algebra II? Steal this sheet.

- **Symbols and expressions.** Polynomials beyond quadratic. Rational expressions rewritten by division. Arithmetic series and a finite geometric series.
- **Linear equations and systems.** Already Algebra I. If they are still fragile, this course is early, not finished.

[209] Hamsa Bastani et al., "Generative AI without Guardrails Can Harm Learning," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. GPT Base: assisted practice +48 percent relative to control; unaided exam −17 percent. GPT Tutor: practice +127 percent; exam statistically indistinguishable from control. Wolfram|Alpha, https://www.wolframalpha.com/, fetched 30 August 2026: computational engine as a check after an unaided attempt.

- **Quadratic equations.** Factor, complete the square, quadratic formula. Real solutions in Algebra I; complex solutions in this course. A year that never wrote $1 \pm i$ has not finished the quadratic.
- **Functions.** Linear, quadratic, polynomial, simple nonlinear, exponential, logarithmic, trigonometric, and fitting models to data. Move among formula, graph, table, verbal. Inverse functions as correspondences, not buttons. A year that never treated a logarithm as the solution of an exponential, and never placed $(\cos\theta, \sin\theta)$ on the unit circle, has not covered school algebra as the Panel defined it.
- **Algebra of polynomials.** Roots, complex numbers, the fundamental theorem as a counting principle, binomial theorem as a named extra.
- **Combinatorics and finite probability.** Counting, independent events, conditional probability continued from Geometry; a survey distinguished from an experiment.

Checklist before moving on, in sitting form.

- Divides a cubic by a linear factor and says the remainder is $p(a)$.
- Writes a complex solution of a quadratic.
- Finds an inverse from a table and says when the swap fails.
- Solves *ab^{ct}* = *d* as a logarithm, and says why ln and $\log_{10}$ differ.
- Places $(\cos 2\pi/3, \sin 2\pi/3)$ without a calculator; uses $\sin^2\theta + \cos^2\theta = 1$ in quadrant II; sketches one period of sine from the circle.
- Says whether a normal model is appropriate, and whether a study was a survey, an experiment, or observational.
- On an unaided mixed page, chooses a family and a method, not only the one in the worksheet title.

What can still be thin: vectors and matrices, polar and parametric, ellipse and hyperbola, inverse trigonometric functions with restricted domains, addition formulas. Those often live in a fourth course. Logarithms as inverses and the unit circle do not.

Transcript. Write **Algebra II** for the third-course catalog. Write **Precalculus** for a fourth course that tightened those functions and picked up a honest share of the extra standards — or **AP Precalculus** if that exam was the object. If the spine was a publisher name, add a parenthetical a stranger can map. Integrated-path families transcript Integrated Mathematics III, or Mathematics III, with a parenthetical a stranger can map onto Algebra II plus remaining geometry. NCAA math core courses are Algebra I or higher; Algebra II sits inside that “or higher.” Named college pages often want three years through Algebra II as a floor.[210] Selective STEM programs will want precalculus or calculus plus an external signal — AP, dual enrollment, a standardized math score. The Panel cited a 1999 federal report for an Algebra II–college-graduation correlation; that is a citation, not a promise that finishing a kitchen Algebra II causes a bachelor's degree.[211]

If the year was leftover quadratics plus a little log, where "a little log" means the LOG button, transcript a more honest name, or finish the catalog. The title is an export object. It has to earn the sheet.

A parent who felt rusty should be able to say, after this chapter: I know why ln and $\log_{10}$ differ. I know the unit circle is why sine is a function of a real number. I know what $\log(a+b) = \log a + \log b$ usually means. I know what to say tomorrow. The child still does the struggle. We can do this.

[210] NCAA Eligibility Center, Home School Toolkit 2025–26: math core Algebra I or higher. Southern Methodist University: suggested Algebra I, Geometry, Algebra II. University of Pittsburgh: minimum three years of mathematics. North Central College: three years including algebra and geometry, one course beyond geometry preferred. Pattern, not a rule.

[211] Clifford Adelman, *Answers in the Toolbox* (1999), cited by NMAP for the Algebra II–college-graduation correlation. Primary not re-opened for this project; cited as the Panel's citation, not as a re-fetched finding and not as a homeschool experiment.

Chapter 7 — Calculus

Figure 8: A paper sketch of a tangent to a curve, a closed laptop, and a paper free-response booklet.

Why this matters

Calculus is the mathematics of instantaneous change and of accumulation. A derivative answers "how fast, right now?" An integral answers "how much has piled up?" The Fundamental Theorem of Calculus is the meeting of those two ideas: the rate at which an accumulated amount is changing is the original rate, and the net change of an antiderivative equals the definite integral.

That is worth the struggle because it is the language of a first college calculus course, of AP Calculus, of many STEM sequences, and of any honest home course that wants to sit beside those objects. Physics that uses derivatives and integrals will expect it. A calculus-based engineering sequence will expect it. A student who wants that path needs this mathematics, not a different senior elective with a similar-looking name.

You do not need calculus for every destination. A student headed toward social science, many biology programs, or civic data literacy may want statistics as the senior college-equivalent, or may want both. Statistics is a different path. It is not a substitute for calculus if the student needs STEM calculus. The next chapter is that path. This chapter is the calculus object.

Senior-year calculus is a pathway fact, not a birthday. On the Common Core map's traditional three-year sequence — Algebra I, Geometry, Algebra II — plus a fourth course, a student reaches precalculus as a junior and calculus as a senior only if high-school content started in Grade 8. Compacted versions of that pathway finish Grade 7, Grade 8, and the first high-school course in two years so calculus is reachable by senior year. Those pathways are models, not mandates. A student who starts Algebra I in Grade 9 finishes Algebra II in Grade 11 and has senior year for precalculus, statistics, or a first calculus course. Trying to do both precalculus and a full BC year in one overloaded fourth course is not a plan.[212]

In 2016, calculus was ever-taught during home instruction to about 15 percent of high-school-level homeschoolers, with a caution on that figure, and taught that

[212] Achieve / CCSS writing team, *Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards*, Oregon ODE copy, https://www.oregon.gov/ode/educator-resources/standards/mathematics/Documents/math-appendix-a-model-course-pathways.pdf, accessed 30 August 2026. Regular three-year pathway plus a fourth course reaches precalculus as a junior and calculus as a senior only if high-school content started in Grade 8. Compacted pathways complete Grade 7, Grade 8, and the first high-school course in Grades 7–8. "The pathways and courses are models, not mandates."

year to about 13 percent of the 9–12 band.[213] That is coverage, not a completion rate, not college calculus, and not a 2026 headcount. No named nationally representative source opened for this project gives the share of homeschool graduates who finish a college calculus course or who major in STEM.[214] Promise a path. A percentile is not the promise.

The door is the function concept.[215] Calculus is a new operation on functions the student already knows. If f assigns to each input exactly one output, then f' is another function — the derivative — that assigns to each input the instantaneous rate at which f is changing there, when that rate exists. The integral, in the accumulation sense, is another operation on f. None of that attaches if f is still a slogan. A student who can differentiate a polynomial by a chant and cannot say, of the same function, what the graph is doing, what the table is doing, and what a verbal description would have to say, has the chant and not the object. Diagnose the door first. Then the eight units of a first calculus course. Then, if those eight are honest, the two extra units that make the course BC.

You do not have to be a mathematician. You do need today's idea well enough to hear "the limit is what you get when you plug it in" as incomplete, and "the derivative is the slope formula" as the Algebra I object wearing a calculus name. That hearing is the job. Teaching every line of BC is a different job. When you are the ceiling for this course, a tutor, a co-op seat, or dual-enrollment Calculus I is a normal high-school plan. The records chapter will say how to put that on paper. This chapter says how to teach the mathematics, or how to know when someone else should.

---

[213] National Center for Education Statistics, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey* (NCES 2020-001), Tables 8 and 9. Calculus ever-taught to 15 percent of high-school-level homeschoolers — interpret with caution — and taught that year to 13 percent of the 9–12 band. Not a completion rate and not college calculus. NHES:2023 did not republish the subject tables. Any tutor, 23 percent; any local homeschool group or co-op, 31 percent.

[214] No named nationally representative source opened for this project gives the share of homeschool graduates who complete a college calculus course or who major in STEM.

[215] Common Core State Standards Initiative, *Common Core State Standards for Mathematics*, 8.F (function concept) and high-school F-IF (function notation, average rate of change). Average rate of change is a difference quotient without the limit.

For the parent: understand it yourself

Many adults feel rusty here. That is ordinary. Give yourself one everyday picture and one precise picture for each of the four objects — limit, derivative, integral, Fundamental Theorem — and five wrong answers you can hear. Then a five-minute warm-up before you sit down.

Everyday picture. A car on a highway. The odometer is accumulation: how far you have gone. The speedometer is an instantaneous rate: how fast you are going right now. Average speed on a trip is distance divided by time — two odometer readings and a clock. Instantaneous speed is what the needle says at one moment. A limit is the approach: as you look at shorter and shorter time intervals around 3:00, the average speeds settle toward the speedometer reading at 3:00, if that reading exists. A hole in a graph is a speedometer that skipped a tick but the nearby averages still settled. A jump is a needle that leaped; the left-hand and right-hand approaches disagree. Filling a tank is the other everyday picture: a flow rate in gallons per minute is a derivative of the volume; the volume added between noon and three is an integral of the flow.

Precise picture. The limit of $f(x)$ as x approaches a is the value that $f(x)$ gets arbitrarily close to as x gets arbitrarily close to a, from both sides if the two-sided limit is the claim, when that value exists. The function does not have to be defined at a. Continuity is the extra claim that it *is* defined at a and that the limit equals the value. The derivative at a is the limit, as h approaches 0, of $[f(a + h) - f(a)] / h$, when that limit exists. That quotient is an average rate — Algebra I already named it. The limit of those average rates is an instantaneous rate. The definite integral of f from a to b is a limit of Riemann sums: slice the interval, take a sample height, multiply height by width, add. As the slices get thinner, when the limit exists, that sum approaches a number. If f is a rate, that number is net change. The Fundamental Theorem, first face: if F is an antiderivative of f, the definite integral from a to b equals $F(b) - F(a)$, under the hypotheses the book names. Second face: if $F(x)$ is the integral of f from a fixed a to a variable x, then $F'(x) = f(x)$. Those are one theorem, read two ways.

You do not need the ε–δ definition this afternoon. You need the picture well enough to hear a collapse.

Wrong answers you should be able to hear.

1. “The limit is the value at the point.” Incomplete even when plugging in

works, because the limit is about approach. Wrong when the point is a hole. A student who always plugs in, including when plugging in is undefined, and then writes "does not exist" without looking at nearby values, has a method and not an object. "The limit does not exist because I got 0/0" confuses an indeterminate form with a limit that may still exist after algebra.

2. "The derivative is the slope formula." The slope formula, $(y_2 - y_1)/(x_2 - x_1)$, is an average rate on an interval. The derivative is a limit of those averages as the interval shrinks to a point. A student who writes "the slope is 2" on a curve, meaning the average on some interval, has mixed the Algebra I object with the calculus object.

3. "It's continuous, so I can differentiate it." Continuity is not enough. A corner, a cusp, a vertical tangent — the derivative fails for different reasons. The absolute-value function at zero is the usual picture: the graph is connected; the secant slopes from the left approach −1 and from the right approach 1; the two-sided limit of the difference quotient does not exist.

4. "The integral is the antiderivative." The definite integral is a limit of sums. The antiderivative is the other face. The Fundamental Theorem is why those two faces meet. A student who can antidifferentiate x^2 and cannot say what a definite integral of a rate means has the chant.

5. "The Fundamental Theorem is $F(b) - F(a)$." Half. The other half is that the derivative of the accumulator is the original function. A kitchen that only computes $F(3) - F(1)$ will meet free-response questions that ask for the shaded region, or for F', or for a sentence about net change.

A sixth, if you have room in the ear: "I don't need the graph; I have the formula." College Board's second mathematical practice for this course is translation across representations. A session that only computes produces a student who can differentiate x^3 and cannot read f' from a graph of f.

The door, before Unit 1. Three questions, not a book title. Can the student move among formula, graph, table, and verbal for a function already on the page? Can the student see $f(g(x))$ as a structure, not as two separate problems? Does the student have unit-circle trigonometry, not only right-triangle SOH-CAH-TOA? "Did we finish a book titled Precalculus" is not among the three. Precalculus is not a legally defined course. A finished book with that word on the spine is evidence of a purchase. Put one function on the table in four representations and stay there until the student can travel. Common gaps at this door, inferred from what the maps require rather than from a kitchen census: a weak function concept; composition not seen as a structure; unit-circle trigonometry never learned; no fluency with exponential and logarithmic functions; a derivative treated as "the slope formula." Those five are the diagnosis list before you buy an AB book, a BC book, a CLEP packet, or a dual-enrollment seat.

College Board's published prerequisite for AP Calculus is the equivalent of four years of college-bound secondary mathematics: algebra, geometry, trigonometry, analytic geometry, and elementary functions — linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piecewise-defined — including properties, composition, algebra, and graphs of functions, and sine and cosine from the unit circle with values at 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ and their multiples. Students who take BC are expected to have basic familiarity with sequences and series and some exposure to parametric and polar equations. CLEP Calculus assumes algebra, geometry, trigonometry, and analytic geometry. A kitchen still teaching the unit circle in week three of AB is doing the door. That is honest work.[216]

Five-minute parent warm-up. Before the lesson, on paper, without a chatbot:

- Write, in one sentence, what a limit is (approach, not plug-in).
- For $f(x) = x^2$, write the average rate of change from $x = 1$ to $x = 1 + h$, then say in words what happens as h goes to 0. (You should get $2 + h$, which approaches 2. That 2 is $f'(1)$.)
- Shade, or imagine shading, the region under $y = 2$ from $x = 0$ to $x = 3$. That area is 6. Say out loud: "the integral of a rate is net change; here the 'rate' is a constant 2, and the accumulation is 6."
- Name today's first problem for the student, and the one question you will ask if they stall.

Check your three sentences against the book's key or, for the derivative computation, against Wolfram|Alpha after you have written yours. Then sit down.

[216] College Board, AP Calculus AB and BC Course and Exam Description, https://apcentral.collegeboard.org/media/pdf/ap-calculus-ab-and-bc-course-and-exam-description.pdf, accessed 30 August 2026, front-matter Prerequisites. CLEP Calculus exam, https://clep.collegeboard.org/clep-exams/calculus, accessed 30 August 2026: assumed after algebra, geometry, trigonometry, and analytic geometry; about 44 questions, about 90 minutes; ACE recommended score 50 for 4 semester hours; list price $97 as of the 30 August 2026 fetch. Two content-split statements on the College Board pages opened that day, both printed in the body.

How to teach it this week

Session shape is the same as the rest of this book: unaided warm-up, short model, student attempt, one good question, mixed practice, exit ticket. Calculus makes the shape stricter because a fluent wrong antiderivative looks like a textbook.

Unaided warm-up (five minutes). Phone in the other room. Three items the student can already attempt: one limit (approach, not only plug-in), one derivative of a composition, one definite-integral interpretation. Example set, labelled as an illustration, not a reported family:

- lim as $x \to 2$ of $(x^2 - 4)/(x - 2)$. Nearby values, then the algebra, then the sentence "the hole is at 2; the limit is 4."
- $d/dx[\sin(3x^2)]$. Composition. Chain rule.
- "The rate at which water enters a tank is $r(t)$ gallons per minute. What does ∫ from 0 to 5 of $r(t)\,dt$ mean?" Net gallons added in five minutes, not "the antiderivative."

Short model. You already did the five-minute warm-up. Teach from the family's book. First pass through a new technique is a worked example, faded. Show a complete solution of one new object — one limit that needs algebra, one chain-rule derivative, one accumulator whose upper limit is not x — with the reason for each line. Then the same structure with a step missing. Then the student generates. Sweller and Cooper's worked-example effect, and Renkl and Atkinson's fading, are the method.[217] They are algebra and skill-acquisition papers, not a calculus trial. Apply the sequence. Fade for accuracy, not for a calendar date.

Exact wording you can steal:

"A limit is what the outputs approach as the inputs get close. Plugging in is a method that works when the function is continuous there. Show me nearby values first."

"Average rate is the slope between two points. Instantaneous rate is the limit of those slopes. Which one is this question asking for?"

[217] John Sweller and Graham A. Cooper, "The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra," *Cognition and Instruction* 2, no. 1 (1985): 59–89. Alexander Renkl and Robert K. Atkinson, "Structuring the Transition from Example Study to Problem Solving in Cognitive Skill Acquisition," *Educational Psychologist* 38 (2003): 15–22. Method applied to a first pass through a calculus technique; not a calculus randomized trial.

"The chain rule is the derivative of the outside, evaluated at the inside, times the derivative of the inside. Is this a composition?"

"The definite integral is a limit of sums. The antiderivative is how we often evaluate it. What does this integral mean in the story?"

Student attempt. Struggle before rescue. Ask, wait, hint, then model. Keep the pencil in their hand. A hint is a question: "Is this a composition?" "Are you looking at an average rate or an instantaneous one?" "Is this a net accumulation, or an area that cannot be negative?" The next step, still not the final expression, is a later hint. The final answer stays off the table until they have produced something.

One good question. After the attempt, one question that forces a second representation. If they computed from a formula, ask for the graph. If they shaded, ask for the sentence. If they wrote $F(3) - F(1)$, ask what F' is.

Mixed practice. After a short blocked introduction to the new move — three chain-rule items, not thirty — mix yesterday's chain rule with last week's implicit derivative and last month's limit that is not a derivative. The student has to choose a strategy, not only execute one. Rohrer, Dedrick, and Burgess found a large benefit for interleaved practice in grade 7.[218] Almost all clean classroom trials of interleaving in mathematics are grade 7. This is a useful study of mixing, not a promise that every home will see the same result on an AP free-response question. Import the shape: blocked to learn execution, mixed to choose.

Exit ticket. Timed, unaided, no model: one mixed three-item check, or a short free-response slice if the student will sit AP. Performance on a page a chatbot helped finish is not learning. Soderstrom and Bjork's review is the reason: current performance is often an unreliable index of lasting change.[219]

First problem for the student, this week, if you are opening limits. Give $f(x) = (x^2 - 1)/(x - 1)$ as a formula, a table of nearby values, and a graph with a hole at $x = 1$. Ask three questions: What is $f(1)$? What is the limit as x approaches 1?

[218] Doug Rohrer, Robert F. Dedrick, and Kaleena Burgess, "The Benefit of Interleaved Mathematics Practice Is Not Limited to Superficially Similar Kinds of Problems," *Psychonomic Bulletin and Review* 21 (2014): 1323–1330. Grade-7 classroom crossover. Almost all clean classroom randomized trials of interleaving in mathematics are grade 7. Shape imported; coefficient not treated as an AP result.

[219] Nicholas C. Soderstrom and Robert A. Bjork, "Learning Versus Performance: An Integrative Review," *Perspectives on Psychological Science* 10, no. 2 (2015). Current performance is often an unreliable index of learning.

Why are those answers allowed to differ? Stay on this function until the student can travel. Then a second function that is continuous, so plugging in works, and the student has to say why.

When to stop talking. After the hint. After the one good question. After you have named the object once in ordinary language. A lecture about the history of Newton and Leibniz is not this week's lesson. The student generates. You hear.

AB, BC, and CLEP, in the week's planning. Decide which object you are teaching before you pace the week.

AP Calculus AB is eight units: Limits and Continuity; Differentiation, definition and fundamental properties; Composite, implicit, and inverse functions; Contextual applications of differentiation; Analytical applications of differentiation; Integration and accumulation of change; Differential equations of the form $dy/dx = ky$; Applications of integration. Multiple-choice weights, from College Board's course page as of 30 August 2026, put the heaviest slices on analytical applications of the derivative and on accumulation (each 15–20 percent). Units 1, 2, 4, and 8 sit at 10–15 percent. Units 3 and 7 sit at 5–10 percent. Those weights are a map of what the exam emphasizes, not a kitchen week-budget.[220]

AP Calculus BC is those eight plus two: parametric equations, polar coordinates, and vector-valued functions; infinite sequences and series. Students who take BC receive an AB subscore. BC is more mathematics, not "harder AB." A course that matched the eight units and stopped is not BC.[221]

CLEP Calculus is a one-semester college-calculus exam: about 44 questions, about 90 minutes, two sections (one with no calculator, one with an online TI-84 Plus CE). College Board describes the content as about 60 percent limits and differential calculus and 40 percent integral calculus on one statement, and as about 10 percent limits, 50 percent differential, and 40 percent integral on another. Both statements were on the pages opened 30 August 2026. Series, polar, parametric, and vector calculus are not this exam. ACE recommends a

[220] College Board, AP Calculus AB course page, https://apcentral.collegeboard.org/courses/ap-calculus-ab, accessed 30 August 2026. Eight units with multiple-choice weightings as given in the body. *AP Calculus AB and BC Course and Exam Description Clarifications and Corrections, Effective Fall 2026*: "Course content has not changed." Mathematical practices as named on the course page.

[221] College Board, AP Calculus BC course page, https://apcentral.collegeboard.org/courses/ap-calculus-bc, accessed 30 August 2026. Units 9 and 10; AB subscore; May 2027 exam timing for both AB and BC as given in the body. Re-check the live exam page after 30 August 2026.

credit-granting score of 50 for four semester hours. Each college sets its own policy. List price on the Calculus page that day was $97; confirm the current fee.[222]

A non-AP home course can match AB content without matching Bluebook, a May sitting, or a graphing-calculator split.[223] It should still match the representations: limit as approach, derivative as a limit of average rates, integral as accumulation, the Fundamental Theorem in two faces, $dy/dx = ky$, area and volume as integrals. Practice the format the student will actually meet. If they will sit AB, write free-response on paper, timed, unaided. If they will sit CLEP, practice a long multiple-choice sitting with two calculator policies. If they will sit dual-enrollment Calculus I, practice what that instructor grades.

Exact sentence for the transcript, said now so the week stays honest. Call the course Calculus, or AP Calculus AB, or AP Calculus BC, or dual-enrollment Calculus I. A stranger should be able to read the line. "Math 12" is not a title. A publisher name without a parenthetical is not a title.

Practice that actually builds learning

Understanding and practice grow together. A new technique needs a short blocked set so the hands learn the move. Then mixed work so the student has to choose. Then a little retrieval of functions, algebra, and unit-circle values already known. Then one incorrect example to diagnose.

Blocked, for the new move. Three to five items of one kind, not a thirty-item page on day one of the chain rule. First chain-rule day: $d/dx[(x^2 + 1)^5]$, $d/dx[\sin(3x)]$,

[222] College Board, AP Calculus AB and BC Course and Exam Description, https://apcentral.collegeboard.org/media/pdf/ap-calculus-ab-and-bc-course-and-exam-description.pdf, accessed 30 August 2026, front-matter Prerequisites. CLEP Calculus exam, https://clep.collegeboard.org/clep-exams/calculus, accessed 30 August 2026: assumed after algebra, geometry, trigonometry, and analytic geometry; about 44 questions, about 90 minutes; ACE recommended score 50 for 4 semester hours; list price $97 as of the 30 August 2026 fetch. Two content-split statements on the College Board pages opened that day, both printed in the body.

[223] Unit-level topic lists below the eight and ten AP Calculus unit titles were not fully extracted from the Course and Exam Description PDF. This chapter stays at unit-title grain. Tall and Vinner 1981 on concept image versus concept definition was not opened; no teaching sequence is invented from it. Non-AP homeschool calculus tables of contents were not opened as primaries.

d/dx[e^{2*x*}]. Same structure. The student executes. You listen for the dropped inner derivative.

Mixed, for when to use it. Tomorrow's page: one chain rule, one product, one limit that is not a derivative, one "what does f' mean on this graph." The student who only ever saw chain-rule items in a chain-rule chapter will treat every composition as a product, or every product as a chain. Mix representations in the same sitting: formula, graph, table, verbal. A *hypothetical* illustration, labelled as such: a parent who runs a page of power-rule items on Monday, a page of graph-reading on Tuesday, and a page of verbal related-rates on Wednesday, and never puts the same function in all three representations on one day, should not be surprised when Thursday's mixed check falls apart. The days of the week are hypothetical. The move is not: one function, four representations, same sitting.

Retrieval. Two minutes. Unit-circle values at the named angles. Derivative of *e^x*, of ln x, of sin and cos. A composition the student already finished last week, regenerated. Fluency here is quick, flexible, accurate recall of the elementary functions calculus will differentiate — not a race, and not optional. Time only facts the student can already get right untimed.

One incorrect example. Build it on purpose. The student explains why the line is wrong, then generates a correct one.

- Wrong derivative: *d/dx*[sin(3x^2)] written as cos(3x^2), missing the chain. Or written as a product.
- Wrong limit: "does not exist because I got 0/0."
- Wrong integral: $F(b) - F(a)$ computed on a function that is not an antiderivative of the integrand, or an indefinite integral with no $+ C$.
- Wrong reading: "it's continuous, so I differentiated it" at a corner.

Booth, Lange, Koedinger, and Newton found that explaining incorrect examples helped conceptual understanding of two-step linear equations in an Algebra I Cognitive Tutor trial.[224] That paper is Algebra I, not calculus. The method transfers as a method: name the error, then generate a correct line. The coefficient does not transfer.

Kitchen, money, and making. A related-rates ladder, a filling tank, a trip's av-

[224] Julie L. Booth, Karin E. Lange, Kenneth R. Koedinger, and Kristie J. Newton, "Using Example Problems to Improve Student Learning in Algebra," *Learning and Instruction* 25 (2013): 24–34. Algebra I Cognitive Tutor. Method transfers as a method.

erage versus instantaneous speed — these can motivate or give a second picture. They do not replace the number line, the written difference quotient, or the unaided problem. The National Mathematics Advisory Panel's Finding 26: real-world contexts improve performance on similar real-world items; they do not, in the small set of high-quality studies the Panel admitted, improve computation, simple word problems, or equation solving. A ladder problem is a second representation of implicit differentiation, not a replacement for it.[225]

Unaided free-response. If the student will sit AP Calculus, every free-response question they will meet is written without a model, on paper, timed. College Board still requires unaided work in the exam room. From May 2027, both AB and BC use a hybrid digital format: multiple-choice and free-response viewed on screen; free-response handwritten in paper booklets. Multiple-choice timing from that sitting: 42 questions in 100 minutes, half the score — Part A, 29 questions, 62 minutes, no calculator; Part B, 13 questions, 38 minutes, graphing calculator required. Free response: 6 questions, 1 hour 30 minutes, half the score. Re-check the live exam page; pages move. Practice the split if that is the sitting. A timed unaided slice is the exam phase applied to this course. Afterward, on your account, you may mark against a rubric you pasted from an official scoring guideline, and you still sample the marking. Models invent justifications and drop minus signs. "Do this FRQ," pasted into a chatbot, is the worksheet with a new font. The student writes. You hold the key.

Four mathematical practices, as College Board names them for this course: determine expressions and values using procedures; translate information across representations; justify reasoning and solutions; use correct notation and conventions (free-response only). A kitchen that only drills procedures is practicing one of four.

For the student

Calculus is a new operation on functions you already know. A function takes an input and gives exactly one output. The derivative is a new function that tells

[225] National Mathematics Advisory Panel, *Foundations for Success: The Final Report* (U.S. Department of Education, 2008), ERIC ED500486. Table 1, Major Topics of School Algebra. Finding 15 (age-lock claims wrong if prerequisites are present). Finding 26 (real-world contexts). Finding 30 (prepared, motivated students may move faster). Recommendation that prepared students have access to authentic algebra by Grade 8.

you how fast the first function is changing at each input, when that rate exists. The integral tells you how much has piled up. Those two operations meet: if you accumulate a rate, the speed at which the pile grows is the original rate. That meeting is the Fundamental Theorem. You will also hear it as "evaluate $F(b) - F(a)$." That is half of it. Hold both halves.

A limit is not "plug it in." A limit is what the outputs get close to as the inputs get close. Sometimes plugging in works, because the function is continuous there. Sometimes there is a hole, and plugging in is undefined, and the nearby outputs still settle. Look at nearby values. Then do the algebra. Then say the sentence.

Tiny worked example. Let $f(x) = x^2$. Average rate of change from 1 to $1 + h$:

$[f(1 + h) - f(1)] / h = [(1 + 2h + h^2) - 1] / h = 2 + h.$

As h goes to 0, that average goes to 2. So $f'(1) = 2$. The graph of $y = x^2$ at $x = 1$ has tangent slope 2. Check: the derivative of x^2 is $2x$, and $2(1) = 2$. Same number, two pictures.

Two tries.

1. Using the same difference-quotient steps, find $f'(3)$ for $f(x) = x^2$. Then check with the power rule. (You should get 6 both ways.)
2. The graph of $y = |x|$ is a V. At $x = 0$, is the function continuous? Is it differentiable? Say why in a sentence. (Connected, yes. Differentiable, no: left-hand secant slopes go to -1, right-hand to 1.)

Explain it back. In your own words, without looking: what is the difference between an average rate of change and a derivative? Give one picture (the car, the graph, or the table) and one formula.

Challenge. $F(x)$ is defined as the integral from 0 to x of $(2t + 1)\,dt$. What is $F'(x)$? What is $F'(3)$? What is $F(3) - F(0)$, and what does that number mean on the graph of $y = 2t + 1$? You should be able to answer the first two without evaluating the integral, because of the second face of the Fundamental Theorem. Then evaluate to check.

You are allowed to struggle. A blank page for two minutes is not failure. A copied fluent paragraph from a chat window is worse than a blank page, because the fluency will convince you that you already know it. Write on paper. Then check.

If you will sit an AP exam, you will write free-response by hand, timed, without a tool. Practice that way. If you will sit CLEP, you will face a long multiple-choice sitting. Practice that way. If you are in a college Calculus I class, practice what that instructor grades. The mathematics — limits, derivatives, integrals, the theorem that joins them — is the same overlapping core. The exam is a different object.

AB is eight units, limits through applications of integration, including simple differential equations $dy/dx = ky$. BC is those eight plus parametric, polar, and vector-valued functions, and infinite series. If your course has not taught series, it is not BC. That is not a judgment of you. It is a title.

If it isn't clicking

Three diagnostics, and the next move for each. No shame. Placement is by skill, not by birthday. A book titled Calculus is a publisher's scope.

1. The algebra and the function concept are the hole, not the limit. Signs: the student can chant a power rule and cannot move among formula, graph, table, and verbal for $y = 2x + 3$; composition looks like two separate problems; trigonometry is only SOH-CAH-TOA; exponential and logarithmic functions are slogans. Next move: close the door. Put one elementary function in four representations every day for a week. Practice $f(g(x))$ as a structure. Rebuild the unit circle at 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$. Slow down. Opening Unit 1 on a fragile function concept produces symbol soup. Going ahead because a neighbor's child is in an AB classroom is not a diagnosis.

2. The object collapsed into a chant. Signs: every limit is a plug-in; every derivative is "the slope formula"; every integral is an antiderivative with no story; $F(b) - F(a)$ is the whole Fundamental Theorem; graphs are skipped because "I have the formula." Next move: one function, four representations, same sitting. Build the incorrect example that matches the collapse — 0/0 declared nonexistent, chain rule dropped, continuity treated as differentiability — and have the student name the error. Then a completion problem (last sentence blank). Then a conventional problem. Fade. If the new technique is the first hypothesis-test analogue in this course — the first chain rule, the first Riemann-sum interpretation — a packet of twenty unworked items on day one is not productive struggle. It is overload. Worked example, faded, then generate.

3. You are the ceiling, and the course has outrun the table. Signs: BC's Units 9 and 10 are on the calendar and series tests are a fog for both of you; every session becomes a lecture you are also learning for the first time; the student is copying worked solutions without generating. Next move: hire a human tutor, join a co-op class with a named instructor, or enroll in dual-enrollment Calculus I. NCES 2016: about 23 percent of homeschoolers had any tutor; about 31 percent had any local homeschool group or co-op; a small share of the 9–12 band also sat college classes. Those are release valves, not a confession. Dual-enrollment Calculus I produces a college transcript a stranger already knows how to read. A kitchen course taught from a textbook can still be an honest Calculus. Hearing a wrong chain rule is a job you can learn. Teaching infinite series from a cold start while also running the rest of the house is a different job.

When to slow down: the door is shut; accuracy on mixed three-item checks is not there; the student cannot say what an integral of a rate means. When to go ahead: the student travels among representations, executes a new technique after a short fade, and can explain a wrong example. When to get a human tutor: Units 9 and 10 without a teacher of record; a parent who cannot yet hear the five wrong sentences and has no week to learn them; a student who needs a college transcript more than a home course. Video curricula exist because many adults feel rusty. Keep yourself as the person who can still hear a wrong derivative. A program that "does 100 percent of the teaching" is a voice in the room. It is not a finding that you can stop hearing.

A *hypothetical* illustration, labelled as such: a parent whose sixteen-year-old has a precalculus book with the last chapter checked, who can differentiate x^3 by a chant, and who, shown a table and asked for the average rate of change, reaches for "the slope formula" as if the table were two points on a line and nothing else, is not ready for Unit 1. The age, the chant, and the checked chapter are hypothetical. The move is: four representations, stay there.

Tools, including AI

Optional helpers for you. The student attempts first. You hold the answer key.

Wolfram|Alpha is a computer-algebra check: verified algorithms, not next-token math.[226] A language model is an explainer for you — why the chain rule wanted that inner derivative, why the Fundamental Theorem has two faces. Mixing them

[226] Wolfram Research, Wolfram|Alpha, https://www.wolframalpha.com/, fetched 30 August 2026. Computational engine on Wolfram Language, not a language model. Check, not explainer.

without a label is how a hallucinated integral survives a glance. After the student attempts a derivative or an antiderivative, Wolfram may check the result. The language model does not sit between the student and $d/dx[\sin(3x^2)]$. Facts and fluent algebra are not a chatbot job. In a large high-school experiment, Bastani and colleagues found that an unguarded GPT-4 that gave answers raised assisted practice grades and then cut unaided exam grades; when asked for the answer, it was fully correct about half the time, with arithmetic errors even when the method was right. Students copied anyway.[227] That study is a classroom trial in Turkey, not a homeschool trial and not a promise about every tool. The mechanism is the transferable piece: access to the solution during practice, copying, a closed test.

Use a tool to explain the idea to you, make extra practice with answers hidden, write a hint script, or diagnose work already done. Never paste the worksheet, never "what's the answer," never photo-to-key, never an unsupervised chatbot during the attempt.

In one box.

- Child attempts first. Parent holds the key.
- Parent-first: "Explain the Fundamental Theorem to me in two faces. Two mistakes a first-year student makes. One tiny problem I should do on paper. Put the answer below PARENT KEY." You solve, then check.
- Extra items: same structure, new numbers. Student sheet and parent key. Five to eight items, not tonight's assigned free-response completed by the model.
- Hint script you read aloud: a question, then the named concept, then the next step — still not the final expression.
- Diagnose a wrong derivative after the attempt: name the misconception; one question to ask; one isomorphic item to retest. The student answers you, not the model.
- After an unaided FRQ: mark against an official rubric you pasted; sample the marking.
- Check a computation with Wolfram, labelled as the check. Check an expla-

[227] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. GPT Base: assisted practice +48 percent relative to control; unaided exam −17 percent. Fully correct 51 percent of the time when asked for the answer; 8 percent arithmetic errors even when the method was right. Classroom RCT in Turkey; not a homeschool study.

nation by whether you can now hear a wrong sentence at the table.

Photomath, Mathway, and Symbolab list calculus among subjects. During the attempt they are the same mechanism as an unguarded answer-giving chatbot. After the attempt, a labelled CAS check is Wolfram or the textbook key.

If Khanmigo is in the house, require the hint dialogue and stay in the room. A Tennessee study of Khan Academy plus Khanmigo found small average gains that resembled Khan Academy practice without the chatbot; most students rarely used the tutor as a tutor. Working paper, not yet journal-peer-reviewed. Expect unused chat unless you require it.[228]

Keep this lane short. The session is still: unaided warm-up, teach, struggle, hint not key, timed unaided slice. “Finish your calculus with a chatbot so we can go to soccer” is the practice-grade illusion.

What “done enough” looks like

Placement by skill, not birthday. A “Calculus” book is a publisher’s scope, not a legal grade. Done enough is a checklist of objects and an honest title.

Before opening Unit 1.

- Moves among formula, graph, table, and verbal for elementary functions already on the page.
- Treats composition as a structure.
- Has unit-circle sine and cosine at the named angles, not only right-triangle trigonometry.
- Fluent enough with exponential and logarithmic functions to differentiate them later.
- Can state average rate of change on an interval and not call it the derivative.

For an honest first calculus course (AB content, or dual-enrollment Calculus I, or a home course that matches those units).

[228] Philip Oreopoulos and Nina Low, NBER Working Paper 35620 (August 2026). Khan Academy plus Khanmigo in Tennessee middle schools. Intention-to-treat about 0.040 population SD per term; gains resemble Khan Academy practice without AI assistance. Working paper, not yet journal-peer-reviewed.

- Limit as approach; one-sided limits; a hole can have a limit; a jump cannot have a two-sided limit.
- Derivative as a limit of average rates; derivative as a function; where f increases, f' is positive.
- Chain, product, quotient; implicit and inverse as far as the course takes them.
- Contextual rate problems and analytical use of f' and f'' (extrema, concavity, optimization), looking at critical points and endpoints on a closed interval.
- Integral as accumulation (Riemann sums in the meaning, not only as a chant); antiderivatives; substitution as the chain rule backward.
- Fundamental Theorem in both faces, formula and graph in the same sitting.
- $dy/dx = ky$, slope fields as a picture, a particular solution from an initial condition.
- Area, volume by slicing, average value — the integral used, sitting on the theorem.
- Mixed three-item checks unaided. If AP is the sitting: timed free-response on paper, unaided.

For BC. Everything above, plus parametric, polar, and vector-valued functions, plus infinite sequences and series (convergence tests as the discipline of saying whether an infinite sum has a sum; Taylor and Maclaurin as approximations). If those two units were not taught, the transcript is not BC. Students who sit BC receive an AB subscore; that is not a reason to sit BC without the extra mathematics.

For CLEP Calculus. The overlapping core: limits, derivatives, integrals, the theorem, $y' = ky$, area. A long multiple-choice sitting, two calculator policies. A score of 50 is ACE's recommended credit-granting score for four semester hours. It is not an AP 4. Search the receiving college. CLEP is prepared for independently; AP follows a designed high-school course and includes free-response. College Board tells families to search each college's policy. Secondary ranking of AP above CLEP is journalism, not a rule in this book.[229]

Credit and seats. AP credit is not automatic. CLEP credit is not automatic. Dual-enrollment credit is not automatic. Each college sets policy. Homeschoolers may self-study AP; a formal AP course is recommended, not required. Parents cannot order AP exams. Find a school through the AP Course Ledger. The coordinator enrolls the student as "exam only." Mid-November order deadline, on the College

[229]College Board, "Comparing CLEP and AP," https://clep.collegeboard.org/for-high-schools/comparing-clep-and-ap, accessed 30 August 2026. College Board tells families to search each college's policy.

Board homeschool guide dated 29 June 2026.[230] Seats are school-controlled. Self-study does not guarantee a May seat. CLEP: register through My CLEP; no school enrollment required. Dual enrollment: the student takes Calculus I as a college student; keep the college transcript; list the course on the homeschool record.

NCAA, in one paragraph, only if you need it. NCAA is a private association's eligibility rules for athletic scholarships, not homeschool law.[231] Mathematics core courses are college-preparatory work at Algebra I or higher. A kitchen calculus course taught from a textbook can count if the worksheet is honest. CLEP is not an NCAA core course. Dual-enrollment Calculus I is a college transcript the association already knows how to read. Evaluation happens after an NCAA school requests it. If athletics are not in view, skip this paragraph. The records chapter has the rest.

Statistics. This course does not substitute for a data-reasoning course. Statistics does not substitute for this course in a calculus-based STEM sequence. College Board's revised AP Statistics framework encourages taking both when the student has the background.[232] Take both if the destination wants both. Take calculus if Physics C or a calculus-based engineering sequence is the destination.

The title on the page. Calculus. Or AP Calculus AB. Or AP Calculus BC. Or Calculus I (dual enrollment), backed by the college transcript. Publisher in a parenthetical or on a description page if you want the book named. Grades and credit a stranger can file. That is done enough as a record. Done enough as mathematics is the checklist above, unaided.

A compacted path through Grades 7–8 exists so calculus is reachable in senior year. It is a model. Compact only if the gates are open: fractions including signed fractions, integer operations, the equal sign as a relation, the function concept, unit-circle trigonometry. The National Mathematics Advisory Panel recommended that prepared students have access to authentic algebra by Grade 8, and found that

[230] College Board, "The Homeschoolers' Guide to College Board Assessments," Jacob Mishook, 29 June 2026. Parents cannot order AP exams; AP Course Ledger; "exam only"; mid-November order deadline; self-study allowed; formal AP course recommended, not required.

[231] NCAA *Home School Toolkit 2025–26* and Core-Course Worksheet (July 2025). Mathematics core is Algebra I or higher. Audited, CLEP, or credit-by-exam courses are not NCAA-approved core courses. Evaluation after an NCAA school requests it. Private association rules, not law.

[232] College Board, *AP Statistics Course and Exam Description*, Effective Fall 2026: non-calculus-based introductory statistics; prerequisite a first-year algebra course. Revised course framework encourages both AP Statistics and AP Calculus when background allows.

claims about children being “too young” for content have consistently been shown to be wrong, and that prepared, motivated students can move faster with no harm to their learning. Neither finding licenses a skip of the function concept because a child is “good at math.” Delay is equally normal. A sixteen-year-old still finishing the door is at a gate, not behind a law.

Chapter 8 — Data and Chance

Figure 9: A scatterplot on paper, a pair of dice, and a sketched histogram.

Why this matters

Statistics is formulating questions, collecting data, analyzing data, and interpreting results, with variability as the central idea rather than "the average." Calculus is instantaneous change and accumulation. Those are two college-equivalents, not one senior elective with two names.

This path matters because data is already on the map from the middle grades, because Algebra I already includes association and residuals if that course was honest, and because a skipped data strand is a retrofit later. It also matters because a student who will live in a world of surveys, medical claims, and "the study shows" needs a way to hear a wrong sentence: "the average is 12, so they are all about 12"; "the correlation is 0.9, so X caused Y"; "I simulated it, so it is true."

AP Statistics, in the Course and Exam Description effective Fall 2026, is equivalent to a one-semester, introductory, non-calculus-based college course in statistics. The published prerequisite is successful completion of a first-year algebra course. The former second-year-algebra gate is gone. That is a fact about this course. It is not a license to stop high-school mathematics after Algebra I, and it is not a claim that statistics covers limits.

College Board's revised framework encourages students who have the background to take both AP Statistics and AP Calculus, and notes that students interested in data science are also encouraged to take AP Statistics and an AP Computer Science course.[233] It is not published as a substitute for calculus in STEM calculus sequences. Choosing statistics *instead of* precalculus or calculus is choosing a different college-equivalent. That is a legitimate choice for a non-STEM path. It is a weak stealth substitute for Calculus I if the destination is Physics C or a calculus-based engineering sequence. Take both if the student has the background and the destination wants both. Take statistics alone if the destination is data literacy, social science, biology, or an introductory-statistics credit the receiving college actually grants.

In 2016, probability was ever-taught to about 24 percent of high-school-level

[233] College Board, *AP Statistics Revised Course Framework*, https://apcentral.collegeboard.org/media/pdf/ap-statistics-revised-course-framework.pdf, partially opened 30 August 2026. Encourages both AP Statistics and AP Calculus when background allows; data-science-interested students also encouraged to take AP Statistics and an AP Computer Science course. Not published as a calculus substitute.

homeschoolers.[234] That is not a statistics-course completion rate and not an AP Statistics sitting rate.[235] Algebra I was ever-taught to 79 percent, geometry to 52, Algebra II to 49, calculus to 15 with a caution. Probability sits below all of them. The data strand is easy to drop because the algebra chapters look like "the real math." This chapter is the refusal to treat that drop as a plan.

You do not have to be a statistician. You do need today's idea well enough to hear "we asked the group, so this is what homeschoolers think" as a sample problem, not a finished claim. A question that has variability is the object. A worksheet of mean, median, and mode with no data story is not.

For the parent: understand it yourself

One everyday picture and one precise picture, then five wrong answers, then a five-minute warm-up.

Everyday picture. The sunflower row in the yard, labelled as an illustration. "How tall is this sunflower?" has one answer. "How tall are the sunflowers in this row?" has many answers. Those answers are a distribution: maybe clustered around one height, maybe pulled by one giant plant, maybe two clumps. The mean is the balance point. The median splits the ordered list in half. The spread says whether they are all near the center or scattered. A survey of "who in this co-op likes the new book" is real data. It is also a volunteer sample of the people who answered, not a simple random sample of U.S. homeschoolers. An observational comparison of two curricula families already chose can show association. It cannot, by itself, justify "the book caused the scores."

Precise picture. A statistical question is a question whose answer varies. A distribution has shape (clusters, peaks, gaps, symmetry or a long tail), center (median, mean), and spread (range is fragile because it is only two points; interquartile range is the middle half; standard deviation is a later, Algebra I, measure of typical distance from the mean). A parameter describes a population. A statistic describes a

[234] National Center for Education Statistics, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey* (NCES 2020-001). Probability ever-taught to 24 percent of high-school-level homeschoolers. Algebra I 79 percent, geometry 52, Algebra II 49, calculus 15 (caution). Any tutor 23 percent; any co-op 31 percent. NHES:2023 did not republish the subject tables.

[235] Homeschool AP Statistics participation rate was not found in a named nationally representative source opened for this project.

sample. Random *sampling* is how a sample can represent a population, with honest uncertainty. Random *assignment* is how treatments can be compared causally, inside the units you actually assigned. Correlation r is a number between -1 and 1 that measures the strength and direction of a linear association. $r = 0.9$ does not mean X causes Y. A residual is the difference between an observed y and the y the line predicts. A small p-value is evidence against a specified null hypothesis, in a specified design. It is not a proof. It is not a cause.

The American Statistical Association's GAISE guidelines, which College Board aligns its four statistical practices to, frame school statistics as formulate, collect, analyze, interpret, with variability at the center.[236] This book treats that spirit. A mean with no question is not the spirit.

Wrong answers you should be able to hear.

1. "The average is 12, so they are all about 12." The list 2, 3, 4, 12, 40 has mean 12.2 and is not "all about 12." Ask for the shape. Two classes can share a mean and not share a spread.

2. "The correlation is 0.9, so X caused Y." Association is not causation by itself. Algebra I already says so. This course's first unit makes the design distinction the slogan does not: random assignment versus random sampling.

3. "We asked the group, so this is the population." A volunteer sample is a biased sample. Who is missing?

4. "p is small, so the null is false and the treatment worked." A small p-value is evidence against a specified null, under the design you actually ran. If the design was observational, a small p-value still does not license a causal sentence.

5. "I simulated it, so it is true." A simulation is a representation. It estimates a probability from a chance process you named. A run count is not a claim until the student writes what was estimated, from what process, in how many trials, compared with what.

A sixth, if you have room: using a proportions procedure on a quantitative mean, or treating paired data as two independent samples. Those are procedure-choice er-

[236] American Statistical Association, *Guidelines for Assessment and Instruction in Statistics Education* (GAISE / GAISE II). Cited inside the AP Statistics CED as alignment for the four practices. This chapter treats the spirit — formulate, collect, analyze, interpret; variability as the central idea — not a quoted page.

rors. Blocked practice teaches execution of one procedure. Mixed practice teaches which one.

The spine already on the map. Statistical thinking does not begin in AP Statistics. Informal data work sits in the early grades: category counts and bars, then line plots of measurements, including fractions on the plot.[237] Grade 6 is the first year the Common Core map names statistical thinking as a critical area: statistical questions; distributions with center, spread, and shape. Grade 7: random sampling; informal comparison of two groups; chance processes, sample spaces, compound events, simulations. Grade 8: scatter plots; informal linear models; clustering, outliers, positive and negative association. An association is not a function until someone specifies a rule, and it is not causation ever, by itself.

Algebra I, if honest, already has shape, center, spread; two-way tables; scatter plots, residuals, correlation versus causation.[238] Geometry already has independence, conditional probability, two-way tables, the addition rule. Algebra II is where the normal model and inference from surveys, experiments, and observational studies live on the traditional pathway. Teach those clusters in the year they sit.[239] This chapter is the through-line and the college-equivalent, not a reason to skip them, and not a dump of AP procedures into Algebra I.

A skipped middle-grades data strand is Unit 1 of AP Statistics as retrofit. NCTM's *Curriculum Focal Points* put analyzing and summarizing data sets among Grade 8's most important topics, and warned that programs offering algebra in Grade 8 should still not omit Grade 7–8 focal-point content.[240] Data is in that warning.

---

[237]Common Core State Standards Initiative, *Common Core State Standards for Mathematics*, https://www.thecorestandards.org/assets/CCSSI_Math%20Standards.pdf, accessed 30 August 2026. Informal data in Grades 1–5; 6.SP, 7.SP, 8.SP; high-school S.ID, S.CP, S.IC as limited to courses in Appendix A. Map, not a homeschool mandate.

[238]Achieve / CCSS writing team, *Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards*, Oregon ODE copy. Algebra I descriptive statistics; Geometry applications of probability; Algebra II normal model and S.IC. "The pathways and courses are models, not mandates."

[239]Integrated Mathematics I–III packages the same clusters across three years. Topic-by-topic Mathematics I from Appendix A pages 49–90 was not extracted in full for this project. Map tonight's lesson onto S.ID, S.CP, or S.IC. Transcript as Integrated Mathematics I, II, or III with a parenthetical a stranger can map.

[240]National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics* Executive Summary (2000): Data Analysis and Probability as one of five content standards across four grade bands. NCTM, *Curriculum Focal Points* (2006): Grade 8 includes analyzing and summarizing data sets; programs offering algebra in Grade 8 should not omit Grade 7–8 focal-

Five-minute parent warm-up. On paper, without a chatbot:

- Write one statistical question you could ask about a week of something you can actually measure (time to complete a warm-up, heights of plants, hours of sleep). Write one question that is *not* statistical because it has a single answer.
- For the list 2, 3, 4, 12, 40, find the median and the mean. Say which number the long tail pulled, and why a news sentence that began "they are all about…" would be wrong.
- One sentence: what is the difference between random sampling and random assignment?
- Name today's first problem for the student, and the one question you will ask if they stall.

Then sit down.

How to teach it this week

Same session shape: unaided warm-up, short model, student attempt, one good question, mixed practice, exit ticket.

Unaided warm-up. Three items: one "census or sample," one correlation-versus-causation, one probability model or "which procedure." Example set, labelled as an illustration:

- A co-op email list asks who likes a new book. Census or sample? Of what population? What bias sits here?
- A scatter of hours of sleep versus quiz scores, $r = 0.6$. Does that license "sleep caused the scores"?
- Two-way table already filled: P(A given B) as a fraction of the B column, not a hunt for the word *given*.

Short model. A question that has variability is the first move. "Find the mean, median, and mode of 3, 5, 5, 7, 12" is a worksheet. It has no question. "How long do our sunflower seedlings grow in a week?" is a question. The answers will differ. The student decides what to measure, on which plants, with which ruler, on which day. Compute the mean after the question, not instead of it.

point content. Focal Points grade lists from a publicly posted excerpt; prefer a library copy of the book.

Exact wording you can steal:

"Is this a question whose answer varies, or a question with one answer?"

"What is the shape? Where is the center? How spread out? Does the mean get pulled by a tail?"

"Was this a sample or a census? Who is missing?"

"Were the treatments assigned at random, or did people choose? Association is allowed. Cause needs a design that can carry it."

"*And* does not always mean multiply. *Or* does not always mean add. *Given* does not always sit next to the condition you should use. What is the chance structure?"

Student attempt. Struggle before rescue. The student writes the sentence. You do not write the AP paragraph for them. Hint is a question: "Is this a sample or a census?" "Were the treatments assigned at random?" Keep the pencil in their hand.

One good question. After a computation, ask for the design. After a design, ask for a representation (dot plot, box plot, two-way table, scatter plus residual sketch). After a simulation, ask for the written claim.

Mixed practice. After a short blocked introduction to one new procedure — three one-sample proportion intervals, not a packet of every test on day one — mix yesterday's proportion interval with last week's one-sample mean test and a "which design is this?" item. Rohrer and colleagues' interleaving trials are grade 7.[241] Import the shape. Leave the coefficient in grade 7. A purchased book that is almost entirely blocked is common. You can still mix two procedures on the same page.

Exit ticket. Unaided: one census-or-sample, one correlation-versus-causation, one "which procedure" or a short written conclusion in context. Phone in the other room.

First problem for the student, this week, if you are opening distributions. Give a short list that is skewed (the 2, 3, 4, 12, 40 illustration, or a list from the book). Ask: What question could this list be answering? What is the shape? Which

[241]Doug Rohrer, Robert F. Dedrick, and Kaleena Burgess, *Psychonomic Bulletin and Review* 21 (2014): grade-7 interleaved practice. Shape imported; coefficient not treated as an AP Statistics result. Doug Rohrer, Robert F. Dedrick, and Marissa K. Hartwig, "The Scarcity of Interleaved Practice in Mathematics Textbooks," *Educational Psychology Review* 32 (2020).

center would you report, and why? What would a sentence that began “they are all about the mean” get wrong? Stay until the student can talk about shape, not only compute.

When to stop talking. After the student has named the design or the chance structure. A lecture about the history of the normal curve is not this week’s lesson.

AP Statistics, five units, in the week’s planning. Effective Fall 2026. First revised exam May 2027. Five units, not the former nine. Journalism about what was “cut” is not the course.[242] The Course and Exam Description is the course.[243]

Unit	Title	Multiple-choice weighting
1	Exploring One-Variable Data and Collecting Data	20–30%
2	Probability, Random Variables, and Probability Distributions	15–25%
3	Inference for Categorical Data: Proportions	15–25%
4	Inference for Quantitative Data: Means	10–20%
5	Regression Analysis	10–20%

Unit 1 is the heaviest slice for a reason: exploring data and collecting it — census versus sample, experiment versus observational study, bias, random assignment versus random sampling — is the load-bearing work, not a warm-up before “the real formulas.” A student who skipped the middle-grades spine and Algebra I’s descriptive statistics will spend this unit learning vocabulary the map already placed. A student who did those years will spend it tightening design.

[242] Journalism claiming five content areas were “cut” from AP Statistics is not used as a primary for what remains. The CED is the primary.

[243] College Board, *AP Statistics Course and Exam Description*, Effective Fall 2026, https://apcentral.collegeboard.org/media/pdf/ap-statistics-course-and-exam-description.pdf, accessed 30 August 2026. Non-calculus-based introductory college statistics; prerequisite successful completion of a first-year algebra course; five units and multiple-choice weights as tabled in the body; four statistical practices aligned to ASA GAISE II. First revised exam May 2027.

Unit 2 makes Geometry's chance work into random variables and named distributions. Expected value is a long-run mean, not a promise about the next trial. Unit 3 is the first inference procedure at AP grain: a confidence interval as an interval of plausible values for a population proportion; a significance test as a measurement of how surprising the sample is if a hypothesized proportion were true. Unit 4 is the same logic for a mean, usually a t-procedure, with one sample, two samples, and paired data as different designs. Unit 5 is regression at AP grain: inference on the slope, residual analysis, extrapolation as a claim to refuse or qualify. Technology computes. The student interprets.

Four statistical practices: formulate questions; collect data; analyze data; interpret results. A unit that is only "compute r" has skipped three of them.

Practice the format the student will actually meet. If they will sit AP, write free-response on paper, timed, unaided. Parents cannot order AP exams; seats are school-controlled; the records chapter has the ledger and the November deadline.[244] If they will sit a dual-enrollment introductory statistics course, practice what that instructor grades. If they will not sit an exam, still teach the spine: a question that has variability, a named design, a chance structure rather than a magic word, a simulation that earns a sentence.

Title on the transcript, said now. Statistics, or AP Statistics, or dual-enrollment Statistics backed by the college transcript. Not "Math 12." Not "data fun." Not a publisher name without a parenthetical a stranger can map.

Practice that actually builds learning

Blocked, for the new move. The first hypothesis-test procedure is high in moving parts: hypotheses, conditions, mechanism, numbers, sentence in context. First: a complete worked example you already understand. Second: the student explains one, including one incorrect example whose error is " p is small, so the null is false and X caused Y." Third: a completion problem with the last sentence blank. Fourth: a conventional problem. Fade. Twenty tests on day one is not a plan.[245]

[244] College Board, "The Homeschoolers' Guide to College Board Assessments," Jacob Mishook, 29 June 2026. Parents cannot order AP exams; AP Course Ledger; "exam only"; mid-November order deadline.

[245] John Sweller and Graham A. Cooper, *Cognition and Instruction* 2, no. 1 (1985); Renkl and Atkinson, *Educational Psychologist* 38 (2003); Julie L. Booth and colleagues, *Learning and Instruction* 25 (2013). Method applied to a first inference procedure; not AP Statistics trials.

The same fade for the first confidence interval, the first two-way table of conditionals, the first residual plot.

Mixed, for when to use it. Which procedure? Which conditions? Which sentence? Proportion versus mean. One sample versus two versus paired. Observational versus experiment. A residual plot that curves is a reason to doubt the linear family, not a vibe.

Retrieval. Two minutes. Median versus mean on a tiny skewed list. P(A given B) from a 2-by-2 table already known. "Association is not causation" said as a design sentence, not a poster: what design would be needed for a causal claim?

One incorrect example, on purpose.

- A volunteer sample labelled "simple random sample."
- A randomized experiment that assigned by who arrived first.
- Conditionals read from the wrong margin of a two-way table.
- A residual plot that fans, with a writer still claiming the linear model is appropriate.
- A confidence interval interpreted as "there is a 95 percent chance that *this* interval contains p," when the course's language is that 95 percent of intervals constructed this way would capture p.

The student names the error. Then generates a better sentence.

Schemas for chance, not keywords. Word problems are types. Chance structures you can name without a new experiment: a single-stage sample space (one spin, one draw); a compound event with replacement or independence (multiply along a tree if independence holds); a compound event without replacement (condition along the tree); a two-way table (joint, marginal, and conditional relative frequencies from the cells); a simulation (repeat a chance process to estimate a probability). IES practice guides reject keyword strategies for word problems.[246] A seventh-grade proportion trial by Jitendra and colleagues is not a probability trial; the teaching move transfers — identify the structure, represent it, solve, check — not the coefficient. *And* does not always mean multiply. Vary the cover story. Never teach a magic word.

Simulation that earns a sentence. Before running it, the student writes what the

[246] Asha K. Jitendra, Jon R. Star, Danielle Dupuis, and Michael C. Rodriguez, *Journal of Research on Educational Effectiveness* 5 (2012): seventh-grade proportion schema trial, not a probability trial. IES/WWC 2021 Recommendation 5 and the 2009 RtI guide reject keyword strategies.

chance process is, what counts as a success, how many trials, and what the estimate will be an estimate *of*. After running it: the count, the estimate, a comparison to any theoretical value the sample space already gave. "I ran it 50 times" is not a claim. "About 18/50 of the trials were both red, so I estimate P(both red) ≈ 0.36, near the 3/8 from the table" is a claim.

Kitchen surveys and kitchen experiments. Real data. They help on similar kitchen-survey items. They do not by themselves teach a confidence interval, a residual plot, or a two-sample procedure the student has not been taught to run. Finding 26's caution still holds as an analogue: similar-context help is not automatic transfer.[247] Use a household scatter as a second representation of a linear association. Name the design. A family that tries two breakfasts on alternate weeks and measures morning-work time has an informal experiment with confounded days, a tiny sample, and no blinding. Use the numbers. Leave them as kitchen data, not as a finished inference unit.

Representations that scale. A two-way table for conditionals, independence, and the addition rule. A dot plot or box plot for shape, center, and spread. A scatter plus a residual plot for association. A tree for staged chance. A simulation for "I cannot list the sample space, but I can estimate." Technology computes a standard deviation the student already understands as spread. It does not introduce spread.

Unaided writing. If the student will sit AP Statistics, every free-response they will meet is written without a model, on paper, timed. Afterward you may mark against an official rubric you pasted, and you still sample the marking. Models invent *p*-values, swap hypotheses, and put causal verbs on observational data. "Analyze this and write the AP paragraph" is the worksheet with a new font.

For the student

Statistics is the mathematics of questions whose answers vary. "How tall is this plant?" has one answer. "How tall are the plants in this row?" has many. Those many answers make a distribution. The distribution has a shape, a center, and a spread. Your job is to ask a real question, get honest data, show the distribution, and write a sentence that does not claim more than the design can carry.

[247]National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 26 (real-world contexts); Table 1 (combinatorics and finite probability as a topic of school algebra through Algebra II).

A sample is not the whole population. People who volunteer to answer are not a random sample. Two things that rise together are associated. Association is not, by itself, a cause. If you want a cause, you need a design that can carry a cause — usually random assignment of treatments, plus comparison, control, and replication. A small p-value is a measurement of surprise if a hypothesized number were true. It is not a proof, and it is not a cause.

Tiny worked example. List: 2, 3, 4, 12, 40. Ordered already. Median: 4 (the middle value). Mean: (2+3+4+12+40)/5 = 12.2. The 40 pulled the mean. A sentence "they are all about 12" is wrong. A better sentence: "The median height (if these were heights in inches of five seedlings) is 4; one plant at 40 pulls the mean to about 12; the distribution is skewed right."

Two tries.

1. A co-op posts a poll in a group chat: "Do you like the new math book?" Twenty families answer, all enthusiastic. Write two sentences: what claim is fair, and what claim is not.
2. A scatter plot of outdoor temperature versus ice-cream sales is strongly positive. Does that mean hot weather causes ice-cream sales? What else might be going on? Write a sentence that uses the word *association* and does not use the word *cause*, unless you can name a design that would support cause.

Explain it back. In your own words: what is the difference between random sampling and random assignment? Why does a homeschool group chat fail at both?

Challenge. You run a simulation of a two-coin toss 40 times to estimate P(both heads). You get both heads 12 times. Write the claim (process, success, trials, estimate). Compare with the sample-space value 1/4. Then write one sentence about whether 12/40 being a little above 0.25 is a problem for the model, or ordinary variability in 40 trials. You do not need a formal test for this challenge. You need a sentence that does not say "I simulated it, so it is true."

You are allowed to struggle. Write the sentence yourself. A copied fluent paragraph about residuals is worse than a blank page. If you will sit AP, you will write by hand, timed, without a tool. Practice that way.

This course is not calculus. Calculus is change and accumulation. You may take both if you have the background. If you need a calculus-based physics or engineering sequence, statistics does not replace that path. If you need to read studies, design a survey, or take a college intro-stats course, this path is the one that names variability on purpose.

If it isn't clicking

1. The question and the design are missing; the formula is not the hole. Signs: every page is mean-median-mode with no story; "we asked the group" is treated as a population; r is computed and then a causal verb appears; a simulation is run and then treated as proof. Next move: stop the worksheet of tidy lists. Ask a question that has variability. Name the design in one sentence before any number. Build the incorrect example that matches the collapse. Slow down on Unit 1 even if the book wants formulas. Going ahead to inference on a student who cannot tell a sample from a census produces fluent nonsense.

2. Chance is a keyword hunt. Signs: *and* always multiplies; *or* always adds; *given* is hunted as a word rather than read from a table; independent means "the problems feel separate." Next move: two-way table on the desk every day for a week. Read joint, marginal, and conditional relative frequencies from the cells. One tree for a staged experiment. Vary the cover story; keep the structure. Geometry's chance chapter, if it was skipped, is the on-ramp. Rebuild it. A first formal probability model is a novice object. Fade it.

3. Procedure choice is a fog, or the course has outrun the table. Signs: every interval is a z for a proportion, including when the data are means; paired data treated as two independent samples; you cannot yet hear the five wrong sentences and the book is in Unit 4. Next move: a mixed three-item "which procedure" page, not twenty more of the same test. If both of you are lost on inference, a human tutor, a co-op statistics class, or dual-enrollment introductory statistics is a normal plan. NCES 2016 already counted tutors and co-ops as ordinary supplements. Dual enrollment produces a college transcript.[248] A video course can supply a voice. You still need to hear " $r = 0.9$, so it caused it."

When to slow down: no statistical question; no design sentence; fractions too shaky to read 0.23 as a relative frequency and as a probability; slope in $y = 2.4x + 11$ not yet "about 2.4 units of y per one unit of x." Those Algebra I objects are

[248] IES/NCEE dual-enrollment outcomes study, first report expected 2027. Homeschool-specific dual-enrollment counts were not in the sources opened. NCES 2020-125 (82 percent of public high schools offered dual enrollment, 2017–18) is public-school supply, not a homeschool rate.

the gate for AP Statistics, not a limit. When to go ahead: the student names the design, chooses a representation, writes a sentence that does not overclaim, and, if inference has begun, chooses the procedure on a mixed page. When to get a human tutor: AP free-response writing that will not form; a parent who needs a week to learn Unit 1 and does not have it; a college intro-stats credit that a community-college course will carry more cleanly than a home sitting.

A *hypothetical* illustration, labelled as such: a parent whose eleven-year-old's Grade 6 book has a statistics chapter after the ratio chapters, who skips it because "the algebra is the real math," and whose fourteen-year-old then meets a scatter plot in Algebra I with no word for association, is meeting a hole the map already named. The skipping and the sitting are hypothetical. The move is: ask a question that has variability in Grade 6, sample in Grade 7, scatter in Grade 8, and do not wait for a course titled Statistics.

AP Statistics after Algebra I is a legitimate senior elective. Algebra II remains useful — the normal model and the inference preview live there — and useful is not required by College Board for this particular exam. Three years of high-school mathematics through Algebra II is still the floor named, as a pattern, on the university pages the records chapter will walk. Statistics on the transcript does not erase that floor. Stopping at Algebra I because "the state doesn't require it" is a college problem whether or not this course is on the page.

Tools, including AI

Optional helpers for you. The student attempts first. You hold the key.

A graphing calculator, a spreadsheet, Wolfram|Alpha, or a statistics package can check a standard deviation, a t-interval, or a least-squares slope *after* the student has named the procedure, the conditions, and the sentence. That is a check, not an explainer.[249] A language model is an explainer for you: why a volunteer sample is biased, why a small p-value is not a cause. Models hallucinate statistics — invented p-values, swapped hypotheses, causal verbs on observational data, residual plots described that were never plotted. If you can hear a wrong answer, you can also hear a fake interval.

Use a tool to explain the idea to you, make extra practice with answers hidden,

[249] Wolfram Research, Wolfram|Alpha, fetched 30 August 2026. Computational check after the student has chosen the procedure.

write a hint script, diagnose work already done, or generate a small table plus a conclusion that is *wrong on purpose* for the student to attack. Never paste the worksheet, never "what's the answer," never photo-to-key, never "analyze this and write the AP paragraph," never an unsupervised chatbot during the attempt.

In one box.

- Child attempts first. Parent holds the key. Procedure choice is the child's. A calculation check comes after.
- Parent-first: "Explain random assignment versus random sampling to me. Two mistakes this age makes. One tiny problem I should do on paper. PARENT KEY below the line."
- Extra items: same structure, new cover story. Student sheet and parent key. Five to eight items, not tonight's free-response completed by the model.
- Hint script you read: "Is this a sample or a census?" then the named concept, then the next step — still not the final interval.
- Diagnose already-wrong work: name the misconception; one question; one isomorphic retest. The student answers you.
- Dataset plus wrong conclusion: twenty rows, one bad sentence (causal claim from an observational scatter, or "they are all about the mean"). Child gets the table and "What is wrong? Write a better sentence." You hold the key.
- Kitchen question design: the model may help you write three statistical questions a week of measurements could answer, and three it could not. You pick. The child measures, plots, writes. The model does not collect, plot, or write the sentence.
- After an unaided FRQ: mark against a pasted official rubric; sample the marking.

Photomath, Mathway, and Symbolab list statistics among subjects. During the attempt they are an answer-giving crutch. A z-score is arithmetic. Expected value is a sum of products. Use a calculator or a package after the student has set up the sum. Use the model to generate a "which procedure" deck, not to sit between the student and the sum.

Session shape is unchanged: unaided warm-up, teach, struggle, hint not key, unaided exit ticket. "Finish your stats with a chatbot so we can go to soccer" is the practice-grade illusion.[250]

[250] Hamsa Bastani and colleagues, *PNAS* 122, no. 26 (2025). Unguarded answer-giving GPT-4: assisted practice up, unaided exam down; fully correct 51 percent of the time when asked for the

What "done enough" looks like

Placement by skill, not birthday. A "grade 6 statistics chapter" is a publisher's scope. Done enough is a spine, then, if you want the college-equivalent, five units and an honest title.

Through the middle grades.

- Asks a question whose answer varies, and can say why "how tall is this plant?" is not one.
- Describes a distribution with shape, center, and spread, and will not say "they are all about the mean" on a skewed list.
- Treats a sample as not a population, and a volunteer sample as biased.
- Reads a scatter: clustering, outliers, positive or negative association; association is not a function until a rule is specified, and not a cause by itself.
- Uses a simulation only with a written claim.

In Algebra I, Geometry, and Algebra II, if those courses were honest.

- Dot plots, histograms, box plots; comparison of two distributions; standard deviation as a named spread.
- Two-way tables; residuals; r; correlation versus causation as a sentence, not a poster.
- Conditional probability from a table; independence checked, not chanted; addition rule as inclusion on a table.
- Normal model as a special shape, not a law of nature; surveys versus experiments versus observational studies named; margin of error as honest width, not a confession.

For an honest Statistics or AP Statistics course.

- Unit 1 design spine: census versus sample; experiment versus observational study; confounding; named sampling methods; named biases; experimental design (comparison, random assignment, replication, control); generalization versus causal inference.
- Probability as random variables and distributions, sitting on Geometry's tables and trees.

answer. Classroom RCT, not a homeschool study. Nicholas C. Soderstrom and Robert A. Bjork, *Perspectives on Psychological Science* 10, no. 2 (2015): performance is often an unreliable index of learning.

- One inference procedure for a proportion, faded, then mixed with others: hypotheses, conditions, numbers, sentence in context. A small p-value does not become a cause.
- The same logic for a mean, with paired versus two-sample as a design distinction.
- Regression: technology computes; the student interprets slope in context, reads a residual plot, refuses silent extrapolation, still will not say $r = 0.9$ is a cause.
- Unaided mixed checks. If AP is the sitting: timed free-response on paper, unaided.

Credit and seats. Each college sets AP and dual-enrollment policy. A 5 is not a registrar's stamp. Search the college. Homeschoolers may self-study; a formal AP course is recommended, not required; parents cannot order exams; find a school through the AP Course Ledger. Dual-enrollment introductory statistics is a college transcript. CLEP is a different object again (credit-by-exam the student can register for without a school); it is not an NCAA core course if athletics are in view.[251]

Not done enough.

- Statistics as a stealth substitute for Calculus I when Physics C or calculus-based engineering is the destination.
- Stopping at Algebra I because this course's official gate is Algebra I.
- Skipping the middle-grades spine because "we'll do AP Stats later."
- Titling the year "Math 12" or "data fun."
- Treating a kitchen poll as a national finding.

Take both calculus and statistics if the student has the background and the destination wants both. That is College Board's own encouragement. This chapter was the data path, from three categories in the early grades to a non-calculus college-equivalent. The student writes the paragraph. You hold the key.

[251] NCAA *Home School Toolkit 2025–26*: CLEP and credit-by-exam courses are not NCAA-approved core courses. Named university pages (Pittsburgh, SMU, North Central, Southwest Baptist) want, as a pattern, about three years of high-school mathematics through Algebra II as a floor. Pattern, not a national rule.

Chapter 9 — Records a Stranger Can Read

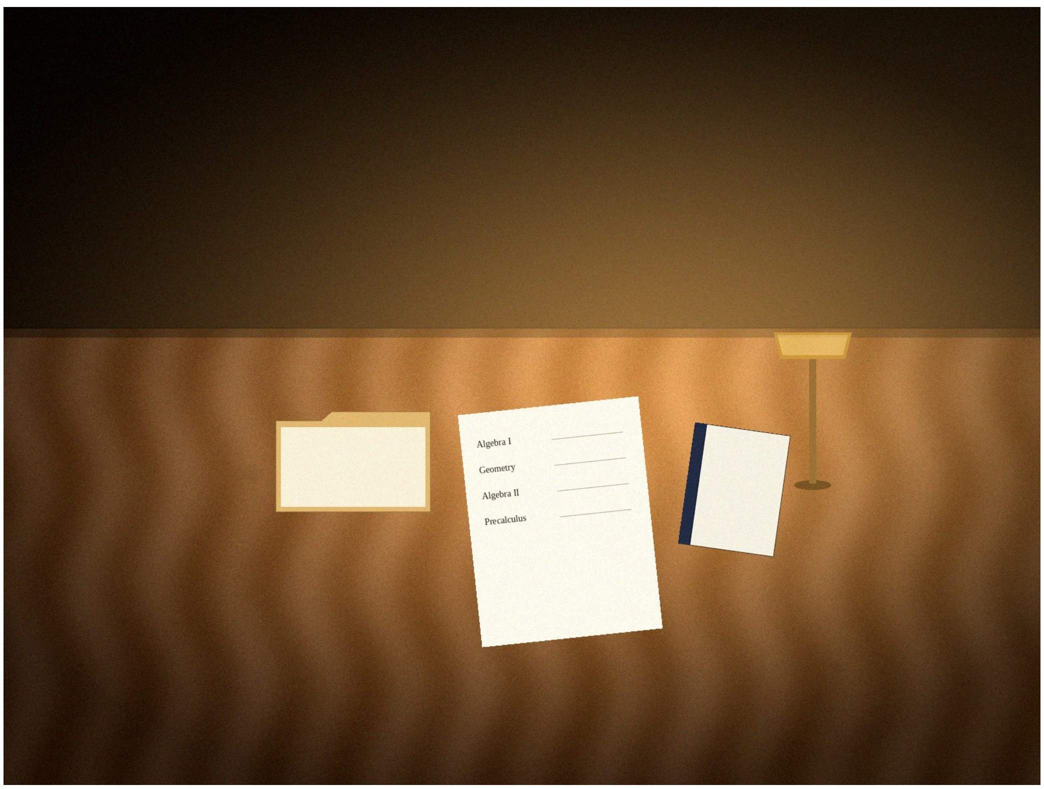

Figure 10: A manila folder, a transcript page with generic course titles, and a plain test booklet.

College-readiness in mathematics is a paperwork job as well as a teaching job. The teaching job was the rest of this book: fractions as a gate, concepts and procedures growing together, the student who struggles before the tool, you hearing a wrong answer. This chapter is the paper trail that work has to leave the house on.

There is no national homeschool diploma. A parent, an umbrella school, a church-school statute, or a private-school affidavit can put a graduation date on a transcript and a signature under it. There is no federal office that converts a satisfied state statute into college evidence. Colleges and, if athletics are in view, the NCAA Eligibility Center will still want a transcript. Some will want course descriptions. Some will want an outside signal — an SAT or ACT score, an AP score, a CLEP transcript, a dual-enrollment transcript. None of those objects is produced by filing a notice, or by not filing one.

State compliance is not college evidence. Some states, Texas among them on the advocacy cards this research opened, can be a legal year of mathematics without a portfolio, without a test, and without a notice, provided a written curriculum includes mathematics. Other states already want a plan, a portfolio, quarterly reports, or a mathematics-inclusive test. Those files are compliance. They are not, by themselves, a high-school transcript a registrar can map onto Algebra I, Geometry, and Algebra II. Read your own state's statute. An advocacy card is a finding aid, not the code.[252] This chapter is not a fifty-state legal manual.

Two readers. Two calendars. A family can satisfy a silent state and still owe a stranger a transcript. A family can satisfy a thick state file and still owe the stranger titles, credits, and grades. An umbrella school can put a name and a signature on the page. It still needs Algebra I written as Algebra I.

Titles a stranger can read

Practice, from the NCAA Eligibility Center's example transcript and from named college pages, is a translation document, not a scrapbook. The fields a stranger can use without a phone call are few:

- Student name and address.
- Ninth-grade start date, month/day/year if athletics are in view.

[252] Home School Legal Defense Association state cards (Texas, Illinois, Idaho, New York, Pennsylvania, North Carolina), accessed 30 August 2026. Advocacy, not the statutes. No national homeschool diploma. Parents read their own state's code.

- Course titles a registrar can map onto Algebra I, Geometry, Algebra II, Precalculus, Calculus, Statistics.
- Grades, credit units, a grading scale, academic year, graduation date.
- Administrator signature.

Dual-enrollment courses listed on the homeschool transcript and backed by the official college transcript. Credits in clean increments: a year of Algebra I is 1.0; a semester is 0.50. NCAA's example treats 1.0 as the maximum on a single line. That increment rule is a useful default even if you will never file an athletic packet.

Publisher level names are not titles. Saxon Math 8/7 is a skill-level book. It is not "eighth grade" and it is not Algebra I. Math-U-See Epsilon is a fractions *level*, not "fifth grade" and not a high-school course. Singapore Dimensions Math 6 is a Grade 6 text with some pre-algebra topics; the publisher says it is not a pre-algebra text in the traditional sense. Beast Academy levels are puzzle-hard elementary and middle-grades books; they are not Geometry.[253] A line that says "Saxon 8/7, A" has told a registrar the publisher and has not told the registrar the mathematics. A line that says "Pre-algebra (Saxon 8/7), A" has done the translation. A line that says "Algebra I, A" with "Saxon Algebra 1, 4th edition" on a course-description page has done it twice, which is what a page like Southern Methodist's will ask for when it wants textbook titles in the descriptions.

Honesty on the title is the same rule as honesty on a worksheet. If the year's work omitted trigonometry and logarithms, Algebra II is not the right title. The National Mathematics Advisory Panel's Algebra II is not "mostly quadratics plus a little log." If the year's work was unfinished arithmetic, Algebra I is not the right title. A beautiful portfolio titled "Math 10" has satisfied one reader and confused the next.

A *hypothetical* illustration, labelled as such: a parent who used Math-U-See Algebra 1, a co-op geometry class, and Teaching Textbooks Algebra 2 writes three lines — "Algebra I (Math-U-See Algebra 1), A, 1.0, 2023–24"; "Geometry (co-op; Prentice Hall), A−, 1.0, 2024–25"; "Algebra II (Teaching Textbooks Algebra 2), B+, 1.0, 2025–26." The course-description page names the texts and the teacher of record. A dual-enrollment College Algebra that spring is listed on the homeschool transcript *and* sent as an official college transcript. The three titles and the sitting

[253] Publisher placement and sequence pages, accessed 30 August 2026: Singapore Math Inc. (Dimensions 6 is a Grade 6 text); HMH / Saxon (8/7 is a skill-level book); Math-U-See (Epsilon is a fractions level); Beast Academy FAQ. Title the high-school course by the mathematics.

are hypothetical. The rule is not.

Placement and acceleration are record-keeping choices as well as teaching choices. Publishers place by skill, not by birthday. A child of ten may place into three different "grades" on three publishers' tests the same afternoon. NCAA and colleges care about the high-school transcript window — ninth-grade start date; Algebra I or higher as core — not about whether Algebra I happened at twelve. Taking Algebra I in eighth grade *and* listing it as a high-school credit is a record-keeping choice with athletic implications. Read the current NCAA Home School Toolkit for pre-ninth coursework rules rather than a forum restatement. Acceleration does not erase a year-count. It changes which titles fill the years. Delay is equally normal. A sixteen-year-old whose transcript still says pre-algebra when the work was pre-algebra has an honest record. A sixteen-year-old whose transcript says Algebra II when the work omitted the functions that make Algebra II is a title problem.

About 12 percent of homeschooled students in 2016 were taught with mostly or strictly informal learning. Lived mathematics does not arrive as "Algebra II, 1.0, B+." A plan, a portfolio, or an athletic worksheet still wants a title and a grading basis after the fact. The translation is possible. It is work. That is a fact about documents, not a verdict on the practice.

Common App is the other translation surface most families will actually meet. Practice, from practitioner guides: the student marks homeschooled; invites the parent, or an umbrella administrator, as counselor; the counselor account uploads the School Report, the transcript, a school profile, and course descriptions. Parent-as-counselor is standard, not a workaround. Confirm against Common App help text in the year you apply.[254] A school profile in this room is a one-page description of the homeschool: who taught, what the grading scale is, whether the school is the kitchen or an umbrella, how dual enrollment and AP were used. It is not a marketing brochure and not a diploma.

A language model may draft a transcript *template* with those fields, or a course-description paragraph that lists the textbook, the major topics, the grading basis, and the credit. You check every line against the actual work. You sign. Asking a model to invent hours, invent a diploma, invent a grading scale the family never used, or back-fill content the course did not cover is a false statement on a document a college will treat as signed. An agent that drafts a template has not signed.

[254]Common App: student marks homeschooled; invites parent or umbrella administrator as counselor. Confirm against live help text in the year of application.

Fabricating a record is not a documentation shortcut.

Algebra I, Geometry, Algebra II — then what the destination wants

Named university pages this research opened want, as a pattern and not as a national rule, about three years of high-school mathematics through Algebra II as a floor.[255]

The University of Pittsburgh's homeschool page: minimum three years of mathematics, plus other core years; a Home School Supplemental Form; a transcript by year with grades; college courses designated and official college transcripts sent; a final transcript from an accredited homeschool organization or signed by the person responsible. Three years is the floor on that page. The page does not, in the text opened, name Algebra I, Geometry, and Algebra II as the three. Filling three years with "Math 9," "Math 10," and "Math 11" meets a year-count and fails the title rule. Filling three years with Algebra I, Geometry, and Algebra II meets both.

Southern Methodist University's homeschool guide: minimum *suggested* three units of mathematics named as Algebra I, Geometry, Algebra II, among other core suggestions; transcript with consistent grading and course descriptions including textbook titles; dual-credit, AP, IB, and historical SAT Subject scores as additional measures in core academic subjects including mathematics. SMU is the page that names the three titles and wants the book on the description.

North Central College: transcript with course names, grades, credit, and year; recommended three years of mathematics including algebra and geometry, one course beyond geometry preferred. Precalculus, Algebra II after a geometry year, or dual-enrollment College Algebra can be that fourth-looking course. Stopping at Algebra I and geometry has not met the preference.

Southwest Baptist University: three units of mathematics among recommended college-preparatory work; parent or organization transcript; ACT, SAT, or the

[255]University of Pittsburgh, Home Schooled Applicants: three years of mathematics. Southern Methodist University homeschool guide: suggested Algebra I, Geometry, Algebra II; textbook titles in descriptions. North Central College: three years including algebra and geometry, one beyond geometry preferred. Southwest Baptist University: three units; ACT, SAT, or CLT. Pattern, not a national rule. A 2026 NACAC homeschool-mathematics PDF was not opened.

Classic Learning Test named; test-optional for admission, with scores that may affect scholarships. Test-optional is not mathematics-optional.

Pattern, not a census, not a NACAC rule — a comprehensive 2026 NACAC policy statement dedicated to homeschool mathematics was not opened as a PDF for this project. Pittsburgh says three years. SMU says three and names the titles. North Central says three including algebra and geometry, one beyond geometry preferred. Southwest Baptist says three units. Selective STEM programs will want precalculus or calculus plus an outside signal — AP, dual enrollment, SAT mathematics. Confirm each destination's current page in the year of application. Pages move.

Stopping at Algebra I because "the state doesn't require it" is a college problem on every named page above. Compliance can sit below that floor. That is the cost of the split this chapter opened with.

In 2016, among high-school-level homeschoolers, Algebra I had been ever-taught during home instruction to 79 percent, geometry to 52 percent, Algebra II to 49 percent, calculus to 15 percent with a caution.[256] Algebra I is common. Algebra II and geometry are not universal. Calculus at home is a minority. The titles have to be earned. They also have to be written down.

Integrated Mathematics I, II, and III is a real packaging. Transcript those names, with a parenthetical a stranger can map onto Algebra I, Geometry, and Algebra II if the receiving reader uses the traditional titles. The mathematics does not change with the cover.

AP seats, CLEP, dual enrollment, SAT — different objects

The documents do not collapse these into one sentence called "outside validation." Each receiving college sets policy. Credit is not automatic for any of them.

SAT. Register online. The homeschool high-school code is 970000.[257] Fee

[256] National Center for Education Statistics, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey* (NCES 2020-001). Ever-taught at high-school level: Algebra I 79 percent, geometry 52, Algebra II 49, calculus 15 (caution). Any tutor 23 percent; any co-op 31 percent; informal learning 12 percent.

[257] College Board, "The Homeschoolers' Guide to College Board Assessments," Jacob Mishook, 29 June 2026. SAT code 970000; PSAT as away student; AP: parents cannot order exams, Course

waivers exist for eligible low-income homeschoolers, directly from College Board. Free practice: Bluebook full-length tests; Official SAT Prep on Khan Academy. The SAT is an admissions test. It is not New York's annual assessment. It is not a diploma.

PSAT. Homeschoolers sit as "away students." Contact a participating school at least four weeks ahead. National Merit eligibility runs only through PSAT/NMSQT. Co-ops may become testing centers. A family that decides in October that a junior should sit the National Merit paper has missed the window if the school's deadline has passed.

ACT is a parallel admissions test. **CLT**, the Classic Learning Test, is accepted at some, often Christian, colleges. Southwest Baptist names it alongside ACT and SAT.

AP. Homeschoolers may self-study. A formal AP course is recommended, not required. Parents cannot order exams. Find a school through the AP Course Ledger. The AP coordinator enrolls the student in an "exam only" My AP section. Mid-November order deadline, on College Board's homeschool guide dated 29 June 2026. Homeschool educators may submit courses to the AP Course Audit to become authorized providers. Seats are school-controlled. Self-study does not guarantee a May seat. A family that finishes a beautiful home AP Calculus AB course in April and has not found a coordinator has a course and does not have an exam. A course labelled "AP" on a homeschool transcript is a different fact from a May score. Title the course Calculus, or AP Calculus AB if the Audit was done, or AP Calculus BC only if Units 9 and 10 were taught. Title statistics Statistics or AP Statistics. A score of 3 or higher may yield credit; each college sets policy. An AP score is not an NCAA core course.

CLEP. Thirty-four subjects, including College Algebra, Precalculus, and Calculus. Register through My CLEP. No school enrollment required. Year-round testing. About 2,900 colleges grant CLEP credit — College Board's figure as of the pages opened 30 August 2026. Each of those colleges still sets its own score and hour policy. ACE's recommended credit-granting score for Calculus is 50 for four semester hours; that is a recommendation, not a transcript line. List price on the Calculus page that day was $97; confirm the current fee. Modern States is named

---

Ledger, "exam only," mid-November deadline; CLEP via My CLEP. CLEP Calculus page, 30 August 2026: ACE 50 / 4 semester hours; list price $97 that day; about 2,900 colleges grant credit (College Board's figure). Search each college's policy.

on College Board pages as a partnership for free prep and an exam voucher; that is College Board's named partnership, not this book's pitch. CLEP is the object a parent can actually order without a school. That is its operational virtue. It cannot do NCAA's job, and it cannot be promised as automatic credit. CLEP is not AP Calculus AB. CLEP is not AP Calculus BC. The overlap in mathematics is real. The assessments are not.

Dual enrollment is the cleanest outside transcript the kitchen can buy when you are the ceiling for the course. The student takes College Algebra, Precalculus, Calculus I, or introductory statistics as a college student. The college sends an official transcript. Admissions offices already know how to file it. NCAA can read it. Pittsburgh's page says college courses should be designated and official college transcripts sent. SMU names dual-credit among additional measures. Cost ranges from free, in some state programs, to ordinary community-college tuition. State dual-credit laws vary. Keep the syllabus. List the course on the homeschool transcript. Send the college transcript.

The 82 percent of public high schools that offered dual enrollment in 2017–18 is public-school supply, not a homeschool rate.[258] Homeschool-specific counts were not in the sources opened. Dual enrollment versus AP versus CLEP are different objects. Journalism that ranks them is journalism. College Board tells families to search each college's policy. If athletics are in view, CLEP is not a core course. If you are the ceiling, dual enrollment is the cleanest of the three as a transcript, because it arrives as a transcript.

NCAA, only if you need it

Skip this section if athletic scholarships at NCAA member schools are not in view.

The National Collegiate Athletic Association is a private association. Its initial-eligibility rules are not homeschool law. They bind athletic scholarships at member schools. The useful sentence for everyone else is already in the title rule: college-preparatory high-school mathematics at Algebra I or higher.

A core course, for the association, is college-preparatory high-school work in English, mathematics (Algebra I or higher), science, social science, world language,

[258] NCES 2020-125: 82 percent of public schools with grades 9–12 offered dual or concurrent enrollment, 2017–18. Public-school supply, not a homeschool rate. IES dual-enrollment outcomes report expected 2027.

comparative religion, or philosophy. Geometry counts. Algebra II counts. Precalculus counts. Calculus counts. Statistics at a college-preparatory high-school level can count if the worksheet supports it. “Saxon 8/7” without a parenthetical does not. “Fun with numbers” does not.

Division I: sixteen core courses, including three years of mathematics at Algebra I or higher; ten of those sixteen, including seven in English, mathematics, or science, before the seventh semester; minimum 2.3 core GPA. Division II: sixteen core courses, including two years of mathematics at Algebra I or higher; minimum 2.2 core GPA. Those year-counts are among sixteen, not instead of sixteen.[259]

The homeschool packet: transcript; an administrator statement about who taught, who graded, who awarded credit, and that the program was conducted in accordance with state law; one Core-Course Worksheet per core course — teacher of record, texts, prerequisites, grading basis, content, signature. Grade and credit on the worksheet must match the transcript. Evaluation happens only after an NCAA school requests it. The association does not pre-approve a kitchen.

Audited, CLEP, and credit-by-exam courses are not NCAA-approved core courses. A student who sits CLEP Calculus instead of taking a year of high-school calculus, and who needs that year as one of three Division I mathematics cores, has bought a college-credit object and has not bought an NCAA core course. A kitchen calculus course taught from a textbook can count if the worksheet is honest and the course is college-preparatory. Dual-enrollment Calculus I is a college transcript the association already knows how to read.

Pre-ninth Algebra I is a Toolkit question. Acceleration does not, by itself, create three years of high-school mathematics. Read the current Toolkit.

What to keep in the file

1. A transcript with titles a stranger can map: Algebra I, Geometry, Algebra II, then Precalculus, Calculus, Statistics as they are earned. Publisher in a

[259]NCAA Eligibility Center, *Homeschool Core-Course Worksheet* (July 2025); *Home School Toolkit 2025–26*; Division I and II initial-eligibility pages. Math core is Algebra I or higher. Division I: 16 core including 3 years math, 2.3 core GPA. Division II: 16 core including 2 years math, 2.2 core GPA. One worksheet per core course. CLEP and credit-by-exam are not NCAA core courses. Evaluation after a school requests it. Private association rules, not law.

parenthetical or on a description page. Credits in clean increments. Your signature.

2. Course descriptions that name the text, the major topics, the grading basis, and who taught. SMU will want the book. Pittsburgh will want a year-by-year transcript. Other pages will want something in between. Write the descriptions once. Reuse them.
3. Dual-enrollment: list the course at home and send the college transcript. Keep the syllabus.
4. AP: the score, if a seat was found. The course titled honestly whether or not a seat was found. Ledger, coordinator, November deadline — start that hunt earlier than April.
5. CLEP: the score report the receiving college asked for, if CLEP is the object that college actually takes.
6. SAT code 970000, or ACT, or CLT if that is the destination. PSAT as an away student with four weeks' lead if National Merit is in view.
7. NCAA packet, only if a coach and a member school are in view: transcript, administrator statement, one worksheet per core course. CLEP is not a core course. Wait for the school's request to evaluate.
8. State compliance in its own folder: notice, portfolio, evaluator letter, required test. That folder is not the transcript.

The paper trail is what the kitchen can actually put in a stranger's hand. Write the titles. Keep the descriptions. Sit an outside signal the destination actually reads. When you are the ceiling, buy a transcript a college already knows how to file. You sign.

Chapter 10 — Programs, Placement, and Extra Help

This book is not a curriculum. Saxon, Singapore, Math-U-See, Beast Academy and Art of Problem Solving, Teaching Textbooks, and Khan Academy are programs you will actually meet. Name them. Compare grain, parent load, and how they place. No homeschool-curriculum randomized trial comparing them was opened for this project, because none was found. Robert Kunzman and Milton Gaither, surveying more than two thousand English-language texts in 2020, note that curriculum research on homeschooling is thin relative to demographics and law.[260] Cathy Duffy Reviews is a long-running homeschool catalogue with "Top Picks," publisher-provided review copies, and affiliate links she discloses. Rainbow Resource is a retailer catalogue.[261] Neither is a trial. Choose by fit: whether you can hear a wrong answer in that program's representations, whether the student will generate the work, special needs, budget, college target. A catalogue star is not a finding.

Pick a tool whose pictures you can learn well enough to notice a wrong turn. Place the student by the publisher's own test, not by birthday. When you are the ceiling for Algebra II through calculus, a tutor, a co-op seat, or a community-college course is a normal high-school plan. NCES 2016 already counted those valves: any tutor, 23 percent; any local homeschool group or co-op, 31 percent; any on-

[260] Robert Kunzman and Milton Gaither, "Homeschooling: A Comprehensive Survey of the Research," *Other Education* 9, no. 1 (2020): curriculum research thin relative to demographics and law. No homeschool-curriculum randomized trial comparing Saxon, Singapore, Math-U-See, Beast Academy, Teaching Textbooks, CTCMath, or Khan Academy was found in the sources opened 30 August 2026.

[261] Cathy Duffy Reviews (catalogue); Rainbow Resource (retailer). RightStart, Math Mammoth, and Life of Fred official scopes not opened as primaries this pass.

line course, 23 percent of homeschoolers and 30 percent of the 9–12 band; a small share of 9–12 also in college classes. Mother was the main provider for 78 percent. Formal curriculum was used by 77 percent.[262] The kitchen is one adult, sometimes supplemented.

Exact list prices move. Confirm on the publisher's page before you buy. Khan Academy remains free.

Six common programs, by fit

Saxon (Houghton Mifflin Harcourt). Incremental and spiral: mixed practice is the point; the publisher tells families not to skip levels; an 80 percent mastery rule of thumb to advance; skill-level placement rather than calendar grade.[263] Split names — Math 5/4, 6/5, 7/6, 8/7 — do not match a birthday. The homeschool sequence HMH publishes runs K, 1, 2, 3, then those split names, then Algebra 1, Algebra 2, Advanced Mathematics, Calculus, with 8/7 or Algebra 1/2 as the pre-algebra fork. Geometry is integrated in some classic editions and also exists as a separate course. Parent load is real on the lesson but lighter than a method you must first learn; the mixed sets do the choosing work this book wants. School-side ESSA evidence claims are about school implementations, not homeschool trials. Transcript the high-school years as Algebra I, Geometry, Algebra II, Precalculus, Calculus — not "Saxon 8/7" without a parenthetical.

Singapore Math Inc. (Dimensions and Primary Mathematics). Concrete–pictorial–abstract, number bonds, bar models, mastery within a level. Parent teaching load is real: bar models are a method you must learn well enough to hear a wrong bar. Dimensions Math 6 "includes some Pre-algebra topics, but is

[262] National Center for Education Statistics, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey* (NCES 2020-001). Main provider mother 78 percent; formal curriculum 77 percent; any tutor 23 percent; any co-op 31 percent; any online course 23 percent (30 percent of 9–12); Algebra I taught that year to 41 percent of the 6–8 band; arithmetic still taught to 31 percent of 9–12. Total hours homeschooled per week in bands, not subject minutes. Informal learning 12 percent.

[263] Houghton Mifflin Harcourt / Saxon Homeschool, *Middle Grades Math Placement Test* and *Saxon Homeschool Math Scope and Sequence*. Skill-level rather than grade-level; score bands into 5/4 through Algebra 1; sequence K–3, 5/4, 6/5, 7/6, 8/7, Algebra 1, Algebra 2, Advanced Mathematics, Calculus. School-line ESSA claims are school implementations, not homeschool trials.

not a Pre-algebra text in the traditional sense; it is a Grade 6 text."[264] Dimensions 7 and 8 together cover pre-algebra and Algebra 1 topics with some geometry; Dimensions 8 expects more prior linear-equation work than a traditional Algebra 1. Editions of Primary Mathematics — US, Standards, 2022, Singapore International — are not interchangeable. Mixing editions is a common hole. Starting at an earlier level is, on the publisher's own page, common and acceptable. A ten-year-old in 3A is a placement result, not a scandal.

Math-U-See (Demme Learning). Mastery with manipulatives — Integer Blocks, "Decimal Street" — video lessons, student-paced, built for the parent who wants a script and blocks. Some materials are lightly Christian, a catalogue observation, not a finding. Thirteen levels: Primer, Alpha, Beta, Gamma, Delta, Epsilon, Zeta, then Pre-Algebra through Calculus. Epsilon is a fractions *level*, not "fifth grade."[265] If a family runs one level per calendar year from Primer, formal fraction operations can sit later than the Common Core Grade 3–5 fractions domain — or earlier, if Alpha started in kindergarten. You are buying a topic, not a grade. A student who cannot yet operate on fractions has not finished Epsilon, whatever the birthday. Parent load is designed: the video and the blocks carry the first explanation; you still need to hear 1/2 + 1/3 = 2/5 as a size mistake.

Beast Academy and Art of Problem Solving. Puzzle-hard. Beast Academy is comic-guide plus practice, Levels 1–5, ages about 6–13, loosely based on the Common Core standards, covering grade-level standards in greater depth. The publisher generally recommends that students start about a level behind their current grade because of problem difficulty; levels 4–5 cover many middle-grades standards.[266] A student who completes Level 5, on the publisher's page, has the background for AoPS Prealgebra. Pedagogically this is the puzzle-hard track, not the default kitchen track. AoPS then offers contest-style texts and classes through calculus. Naming that pipeline is not a claim that most homeschoolers are shopping

[264] Singapore Math Inc., School FAQ and Placement Tests pages, accessed 30 August 2026. Dimensions Math 6 is a Grade 6 text, not a traditional pre-algebra text; Dimensions 7 and 8 together cover pre-algebra and Algebra 1 topics with some geometry. Untimed PDFs; starting at an earlier level common and acceptable; editions not interchangeable.

[265] Math-U-See / Demme Learning, curriculum sequence and placement tool, accessed 30 August 2026. Thirteen levels, Primer through Calculus. Epsilon is fractions. Placement: yes/no mastery items; scoring guide in the 80–89 percent range to advance.

[266] Art of Problem Solving, Beast Academy books and FAQ, accessed 30 August 2026. "We generally recommend that students start about a level 'behind' their current grade level." Levels 4–5 cover many middle-grades standards. Level 5 as background for AoPS Prealgebra is publisher language. Pricing table did not render a public dollar table on the 30 August 2026 fetch.

for it. Parent load is high on the mathematics even when the comics look friendly. A ten-year-old is not placed into Level 5 because a neighbor's ten-year-old uses it.

Teaching Textbooks. An app that, in the publisher's own words on the homepage this research opened, "does 100% of the teaching" and "100% of the grading,"[267] with audiovisual solutions and a tutor helpline. That is the publisher's claim, not an effectiveness trial, and not a finding that you can stop hearing wrong answers. Catalogue consensus, not a study, describes gentler pacing, sometimes behind typical public-school grade labels. Parent load is light on explanation and still real on supervision: the student generates; you listen for a magnitude error, an operational equal sign, a derivative treated as the slope formula. The official scope-and-sequence PDF was not opened for this project; grade-by-grade topic lists from listicles stay out of this book. List price was not displayed on the homepage on 30 August 2026; third-party figures circulate and were not confirmed on the primary.

Khan Academy. A nonprofit, free mastery-practice map, including SAT practice in partnership with College Board. Useful as practice and as a free map of AP Calculus AB and BC. Not a complete homeschool program of record unless you build the record: titles, grades, credits, a description of what was actually done. Khanmigo is a paid, constrained tutor over Khan content; "never gives you the answer" is publisher copy, not a trial. A Tennessee study of Khan Academy plus Khanmigo found small average gains that resembled Khan Academy practice without the chatbot; most students rarely used the tutor as a tutor. Working paper, not yet journal-peer-reviewed.[268] If Khanmigo is in the house, require the hint dialogue and stay in the room. Khan Academy does not, by itself, produce Algebra I on a transcript.

Others you will see named. CTCMath is a video-plus-practice system a catalogue reviewer treats as a homeschool option, with diagnostic tests by topic and a family plan that unlocks all levels. RightStart, Math Mammoth, and Life of Fred appear in homeschool roundups; their official scopes were not opened as primaries for this project and are not promoted here to handbook fact. Public-school virtual and

[267] Teaching Textbooks homepage, https://www.teachingtextbooks.com/, accessed 30 August 2026. Claims "100% of the teaching" and "100% of the grading"; tutor helpline. Not an effectiveness trial. List price not displayed on the homepage that day.

[268] Khan Academy; Khanmigo pricing page, accessed 30 August 2026. Khan Academy content free. Khanmigo $4/month or $44/year on the page that day; teachers free. "Never gives you the answer" is publisher copy. Philip Oreopoulos and Nina Low, NBER Working Paper 35620 (August 2026): Khan Academy plus Khanmigo in Tennessee middle schools; gains resemble Khan Academy practice without AI assistance. Working paper.

charter-at-home are often public-school enrollments with a teacher of record, state tests, and an IEP if eligible. They are not the parent-as-teacher kitchen. Co-ops, umbrella schools, cottage schools, and microschools are legal and social wrappers. Umbrella schools can issue transcripts. They do not magically teach calculus.

NCES 2016 reports total hours homeschooled per week, not subject minutes. There is no national study in the sources opened for typical daily mathematics minutes. Use a sitting the student can finish.

Placement tests exist because the labels lie

A “grade 6” book is a publisher’s scope, not a legal grade. A child of ten may place into Singapore 3A, Saxon 6/5, and Math-U-See Gamma on the same afternoon. That is not a contradiction. It is three grain-sizes of “grade.” In 2016, Algebra I was taught that year to 41 percent of the homeschooled 6–8 band; arithmetic was still taught to 31 percent of the 9–12 band. Grade labels in this book are sequence anchors, not prison bars. Homeschool law rarely forbids a twelve-year-old from taking geometry or a sixteen-year-old from finishing fractions.

Saxon. Middle-grades placement PDF: score bands into Math 5/4, 6/5, 7/6, 8/7, Algebra 1/2, or Algebra 1. The publisher says the test is not infallible; also consider age, prior curriculum, and how quickly the student grasps concepts. Skipping a split name because it looks like a grade you want to leave is how a hole in fractions or signed numbers arrives at Algebra I.

Singapore. Free untimed PDFs by series and edition, about an hour. The publisher instructs parents not to use the test as the sole factor and treats starting at an earlier level as common and acceptable. Watch how the student works, not only the score. Use the test for the edition you will actually buy.

Math-U-See. Interactive yes/no mastery questionnaire plus a printable pretest. Scoring materials use a mastery threshold in the 80–89 percent range to advance. The question is “has this topic been mastered,” not “what grade is this child.”

Beast Academy. Age-based first guess, then unit pretests. Honor the publisher’s “about a level behind” sentence. The comics looking friendly is not a placement.

Teaching Textbooks. Place with the publisher’s own tools if you choose that app. Topic lists from blogs are not a placement.

Khan Academy. Course lists and unit quizzes can show what a student can already do on that map. They do not replace a publisher's placement if you are about to buy a print sequence, and they do not by themselves write Algebra I on a transcript.

The National Mathematics Advisory Panel: what is developmentally appropriate depends mainly on prior opportunity to learn, not on being "too young"; prepared, motivated students may move faster. Accelerate a child who has the prerequisites. Delay a child who does not. Skipping fractions because a child is "good at math" is not acceleration. A child may enter homeschool missing a gate. The first job is diagnosis.[269]

New York plans and some dual-enrollment offices still want a grade. Pennsylvania tests in grades 3, 5, and 8. NCAA wants a ninth-grade start date. You can organize by mathematics *and* still export a grade-labelled transcript. The records chapter is that export. This chapter's job is to let the placement test speak before the birthday does.

Before Algebra I, diagnose fractions including signed fractions, integer operations, and the equal sign as "the same amount." Before calculus, diagnose four representations of a function, composition as a structure, and unit-circle trigonometry. A finished book titled Precalculus is a purchase, not that diagnosis.

When a tutor or a community-college course helps

You do not have to be a mathematician. You do need today's idea well enough to hear a wrong answer. When that hearing is not yet available for this year's course, extra help is ordinary.

A weekly tutor is help. Unless the tutor's institution issues a transcript, you remain the grader and the administrator of record. Name the tutor on a course description if you want the help visible. A co-op class with a named instructor, a named text, and a grade the instructor assigned is easier to describe on a transcript or an athletic worksheet than a year you are not sure how to narrate. You still sign as administrator unless an umbrella school does.

Dual enrollment is a college course that produces a college transcript. College Algebra, Precalculus, Calculus I, and introductory statistics are the usual mathemat-

[269] National Mathematics Advisory Panel, *Foundations for Success* (2008), Findings 15 and 30. NCES 2020-125 dual-enrollment figure is public-school supply.

ics objects. Admissions offices already know how to file that transcript. NCAA can read it. Cost ranges from free, in some state programs, to ordinary community-college tuition. State dual-credit laws vary. Keep the syllabus. List the course at home. Send the college transcript. Homeschool-specific dual-enrollment counts were not in the sources opened; the federal 82 percent figure is public-school supply. A community-college seat is a release valve, not a claim about what most homeschoolers do.

A *hypothetical* illustration, labelled as such: a parent who last took algebra years ago hires a weekly tutor for Algebra II, or enrolls the student in College Algebra at the community college, and remains the homeschool administrator of record. The age of the last algebra course is hypothetical. The 23 percent / 31 percent tables are not. That plan is a normal high-school year.

Video curricula exist because many adults feel rusty. Keep yourself as the person who can still hear a wrong inverse, a dropped chain rule, or "$r = 0.9$, so it caused it." Handing the same student an unconstrained chatbot is not the same valve. The teaching chapters already named the split: the student attempts first; you hold the key; Wolfram checks a computation after the attempt; a language model may explain an idea to you. Photomath, Mathway, and Symbolab will print a path. During the attempt they are a crutch.

Public-school à la carte access is state-law dependent. An umbrella school's transcript still needs titles a stranger can map. Special education at home is not an IEP with a Free Appropriate Public Education duty in the way public school is. If you are the ceiling for a learner with a mathematics disability, buy expertise rather than an AI workaround.

A short chooser, not a ranking

- If you want mixed daily practice and can live with spiral review, Saxon is built that way. Place by the score-band test. Leave every split name in place; chasing a birthday is how a hole arrives.
- If you want bar models and a mastery level, and you are willing to learn the bars, Singapore is built that way. Place with the untimed PDF for the edition you will use. Starting earlier in the sequence is expected.
- If you want blocks, a video, and a topic name (Epsilon is fractions), Math-U-See is built that way. Place by mastery items. Advance when the topic is

there.
- If the student thrives on hard puzzles and you can hear wrong answers in that style, Beast Academy then AoPS is built that way. Start about a level behind the birthday.
- If you need a voice that presents and grades, Teaching Textbooks is built that way. You still listen.
- If you need free practice and a map, Khan Academy is built that way. You still write the record.

None of those bullets is “best.” Fit is whether you can hear a wrong answer in that program’s representations, and whether the student will generate the work. When the student has unfinished fractions, go back, whatever the calendar says. When you are the ceiling, buy a tutor, a co-op seat, or a community-college transcript. When the destination is a stranger with a transcript in hand, title the year Algebra I, Geometry, Algebra II, Precalculus, Calculus, or Statistics — the mathematics, not the franchise.

A Note on Sources

Studies named in the chapters are listed in Notes, in one series at the back. That is where the full citations live, so the teaching pages can stay a teaching voice.

Some items the research behind this book did not open stay out of the teaching voice. I did not invent a coefficient, a price, or a product feature to fill a hole. If a study is in the chapter, it is in the notes. If we could not open it, it is not used as a finding here.

The bibliography that follows is the working list from that research. Access date 30 August 2026.

Bibliography

Mathematics for Homeschooling Michael Gannotti
Chicago Manual of Style, 17th edition, notes-bibliography system

Works named in the notes, plus the research list compiled 30 August 2026. Access date 30 August 2026 unless a later fetch is noted. Status tags after an entry ([Cited], [Consulted], [Vendor], [UNVERIFIED], and the rest) mark how the item was used in the research, not a ranking.

---

1. Standards, sequences, and college-credit maps

Achieve in partnership with the Common Core State Standards writing team. *Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards*. 2010. Oregon Department of Education hosting copy, https://www.oregon.gov/ode/educator-resources/standards/mathematics/Documents/math-appendix-a-model-course-pathways.pdf. Accessed 30 August 2026. Pathways (Traditional, Integrated, compacted) are "models, not mandates." Geometry six critical areas; G.CO.9–11 instructional note quoted in full against this copy, 30 August 2026. Integrated Mathematics I–III topic-by-topic (pp. 49–90) only partly extracted. [Cited]

Achieve. *The Model Course Mathematics Pathways*. 2010. https://www.achieve.org/files/TheModelCours
Accessed 30 August 2026. Overview of Traditional versus Integrated high-school pathways aligned to CCSSM Appendix A. Full report opened via search, 30 August 2026. [Consulted]

ACT and The Education Trust. *On Course for Success*. 2005. Cited inside CCSSM Appendix A as empirical grounding that the Algebra I–Geometry–Algebra

II topic set matches high-poverty schools where students met ACT benchmarks. Full report not re-opened this pass. [Consulted; UNVERIFIED full text]

Adelman, Clifford. *Answers in the Toolbox: Academic Intensity, Attendance Patterns, and Bachelor's Degree Attainment*. Washington, DC: U.S. Department of Education, 1999. https://eric.ed.gov/?id=ED431363. Accessed 30 August 2026. Cited by the National Mathematics Advisory Panel for the Algebra II–college-graduation correlation. Correlation, not causation. Primary not re-opened this pass; cited as the Panel's citation. [Cited; UNVERIFIED full text this pass]

American Statistical Association. *Guidelines for Assessment and Instruction in Statistics Education*. Pre-K–12 GAISE I, 2005 (rev. 2007); Pre-K–12 GAISE II, 2020. https://www.amstat.org/education/guidelines-for-assessment-and-instruction-in-statistics-education-(gaise)-reports. Accessed 30 August 2026. The 2016 date belongs to the *college* GAISE report revision, not the school document (mid-2 errata). AP Statistics CED aligns its practices to GAISE. ASA primary not opened in full; page-level claims UNVERIFIED. [Cited; UNVERIFIED page-level]

California Department of Education. *California Common Core State Standards: Mathematics*. Electronic edition. 2013. https://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug Accessed 30 August 2026. State edition of CCSSM plus California additions (marked CA) and model high-school courses. Do not treat CA additions as national CCSSM. [Cited]

College Board. "AP Calculus AB Course." AP Central. https://apcentral.collegeboard.org/courses/ap-calculus-ab. Accessed 30 August 2026. Eight units; 2026–27 clarifications do not change course content. MC unit weights from the official course page. [Cited]

College Board. "AP Calculus BC Course." AP Central. https://apcentral.collegeboard.org/courses/ap-calculus-bc. Accessed 30 August 2026. The eight AB units plus Unit 9 (parametric, polar, and vector functions) and Unit 10 (infinite sequences and series). BC students receive an AB subscore. [Cited]

College Board. "AP Calculus BC Exam." AP Central. https://apcentral.collegeboard.org/courses/ap-calculus-bc/exam. Accessed 30 August 2026. MC timing change effective May 2027: 42 MC / 100 minutes (Part A 29 / 62 no calculator; Part B 13 / 38 graphing calculator required). FRQ 6 questions / 1 hour 30 minutes / 50%; hybrid digital; FRQ handwritten. [Cited]

College Board. *AP Calculus AB and BC Course and Exam Description*.

https://apcentral.collegeboard.org/media/pdf/ap-calculus-ab-and-bc-course-and-exam-description.pdf. Accessed 30 August 2026. Front-matter "Prerequisites" extracted 30 August 2026 (end-errata): four years of college-bound secondary mathematics; algebra, geometry, trigonometry, analytic geometry, and elementary functions including inverse trigonometric and piecewise-defined; properties, composition, algebra, and graphs of functions; unit-circle sine and cosine at 0, π/6, π/4, π/3, π/2 and multiples; BC extra: basic familiarity with sequences and series and some exposure to parametric and polar equations. Unit-guide topic rows below the eight and ten titles remain UNVERIFIED. [Cited]

College Board. *AP Calculus AB and BC Course and Exam Description Clarifications and Corrections, Effective Fall 2026*. https://apcentral.collegeboard.org/media/pdf/ap-calculus-ab-bc-course-and-exam-description-clarifications.pdf. Accessed 30 August 2026. EVT wording; infinitely many DE solutions; MC timing. Course content unchanged. [Cited]

College Board. "AP Precalculus Course." AP Central. https://apcentral.collegeboard.org/courses/ap-precalculus. Accessed 30 August 2026. Official course-page unit table (end-errata): Unit 1 Polynomial and Rational Functions 30%–40% MC; Unit 2 Exponential and Logarithmic Functions 25%–40%; Unit 3 Trigonometric and Polar Functions 30%–35%; Unit 4 Functions Involving Parameters, Vectors, and Matrices not assessed. Content unchanged in 2026–27 clarifications; exam format changes May 2027. Topic lists below those titles UNVERIFIED (full CED not opened). [Cited]

College Board. *AP Statistics Course and Exam Description*. Effective Fall 2026. https://apcentral.collegeboard.org/media/pdf/ap-statistics-course-and-exam-description.pdf. Accessed 30 August 2026. Five units; "equivalent to a one-semester, introductory, non calculus-based college course in statistics." Prerequisite: successful completion of a first-year algebra course. [Cited]

College Board. "AP Statistics Course." AP Central. https://apcentral.collegeboard.org/courses/ap-statistics. Accessed 30 August 2026. Companion to the Fall 2026 CED. [Cited]

College Board. *AP Statistics Revised Course Framework*. https://apcentral.collegeboard.org/media/pdf/ap-statistics-revised-course-framework.pdf. Accessed 30 August 2026. Encourages taking both AP Statistics and AP Calculus when background allows. Not published as a STEM-calculus substitute. Partially opened 30 August 2026. [Cited]

College Board. "Do I need to enroll in a class section in My AP if I'm homeschooled?" AP Students Help Center. https://apstudents.collegeboard.org/help-center/do-i-need-enroll-class-section-my-ap-if-im-homeschooled. Accessed 30 August 2026. Homeschoolers cannot order AP Exams; AP Course Ledger; exam-only My AP section; mid-November order deadline. [Cited]

College Board. "Special Score Structure: Calculus BC." AP Students. https://apstudents.collegeboard.org/about-ap-scores/special-score-structure-calculus-bc. Accessed 30 August 2026. BC students receive an AB subscore. Official page refetched 30 August 2026 (end-errata). [Cited]

College Board / ETS. *CLEP Calculus Test Information Guide, 2015–16*. Hosted copy, College of the Mainland, https://build.com.edu/uploads/sitecontent/files/testing-center/MATH2413_Calculus_I.pdf. Older information guide; topic list consistent with the live 2026 page. Prefer the live page. [Consulted]

College Board. "CLEP Calculus Exam." https://clep.collegeboard.org/clep-exams/calculus. Accessed 30 August 2026. One-semester college calculus; one statement ~60% limits and differential / 40% integral; another ~10% / ~50% / ~40%. 44 questions; ~90 minutes. ACE recommended score 50 / 4 semester hours; each institution sets policy. List price $97 this access day. [Cited]

College Board. "Comparing CLEP and AP." https://clep.collegeboard.org/for-high-schools/comparing-clep-and-ap. Accessed 30 August 2026. Independent prep versus designed high-school course; credit policies set by colleges. [Cited]

Common Core State Standards Initiative. "Grade 7 Introduction." https://www.thecorestandards.org/Math
Accessed 30 August 2026. Official HTML restatement of Grade 7 critical areas, including inferences from samples. [Cited]

Common Core State Standards Initiative. "Grade 8 Introduction." https://www.thecorestandards.org/Math
Accessed 30 August 2026. Official HTML restatement of Grade 8 critical areas, including bivariate data. [Cited]

National Governors Association Center for Best Practices and Council of Chief State School Officers. *Common Core State Standards for Mathematics*. 2010. https://www.thecorestandards.org/assets/CCSSI_Math%20Standards.pdf. Accessed 30 August 2026. "Do not dictate curriculum or teaching methods." K–8 grade-by-grade map; high-school conceptual categories. Map, not a homeschool mandate. A 2026 state-by-state census of who still uses CCSSM is UNVERIFIED; do not say "most states." [Cited]

U.S. Department of Education. "Fact Sheet — National Math Panel Final Report."

2008. https://www.ed.gov/media/document/factsheetpdf-89274.pdf. Accessed 30 August 2026. Official restatement of NMAP findings on fractions and Major Topics of School Algebra. [Consulted]

IEA TIMSS 2015 Encyclopedia. "The Mathematics Curriculum in Primary and Lower Secondary Grades" (Singapore). 2016. https://timss2015.org/encyclopedia/countries/singapore/the-mathematics-curriculum-in-primary-and-lower-secondary-grades/. Accessed 30 August 2026. Official-curriculum summary through Grade 8, including linear and quadratic functions/graphs by Secondary 2. [Cited]

Institute of Education Sciences / NCEE. "Outcomes Associated with Dual Enrollment Programs." Project page. https://ies.ed.gov/use-work/evaluations/outcomes-associated-dual-enrollment-programs. Accessed 30 August 2026. Cites ~2.5 million dual-enrolled students in 2022–23; first study report expected 2027. Not yet an outcomes paper. [Cited]

McCallum, Bill, and the CCSSM Progressions authors. *Progressions for the Common Core State Standards for Mathematics*. Compiled 28 February 2023. https://mathematicalmusings.org/wp-content/uploads/2023/02/Progressions.pdf. Accessed 30 August 2026. Authoritative unofficial compilation of the draft progressions that informed CCSSM. Full PDF not line-read this pass. [Consulted; UNVERIFIED full line-read]

Ministry of Education, Singapore. *G2 and G3 Mathematics Syllabuses (Secondary One to Four)*. 2020. https://www.moe.gov.sg/-/media/files/secondary/fsbb/syllabus/2020-g2-and-g3-mathematics-syllabuses.pdf. Accessed 30 August 2026. Full SBB secondary syllabuses; implementation from 2020 Secondary One cohort. Partially opened 30 August 2026. [Cited]

Mishook, Jacob. "The Homeschoolers' Guide to College Board Assessments." College Board All Access, 29 June 2026. https://allaccess.collegeboard.org/homeschoolers-guide-college-board-assessments. Accessed 30 August 2026. SAT homeschool high-school code 970000; PSAT as away student; AP exam-only via school coordinator; CLEP self-registration; fee waivers. Association policy communication, not a college. [Cited]

NCAA Eligibility Center. *Homeschool Core-Course Worksheet*. July 2025. http://fs.ncaa.org/Docs/eligibility_center/Student_Resources/Core_Course_Worksheet.pdf. Accessed 30 August 2026. Teacher of record, texts, prerequisites, grading basis, parent/administrator signature. Grade/credit must match transcript. [Cited]

NCAA Eligibility Center. *Home School Email Cover Sheet*. Updated January 2023. http://fs.ncaa.org/Docs/eligibility_center/Student_Resources/Cover_Sheet.pdf. Accessed 30 August 2026. Submission checklist: transcript, administrator statement, worksheets, proof of graduation. Still linked from the 2025–26 toolkit. [Cited]

National Collegiate Athletic Association. "Division I Initial Eligibility Requirements." https://www.ncaa.org/eligibility-center/initial-eligibility-requirements/division-i/. Accessed 30 August 2026. 16 core courses including 3 years mathematics (Algebra I or higher); 10/7 progression; 2.3 core GPA. Association rules, not a statute. [Cited]

National Collegiate Athletic Association. "Division II Initial Eligibility Requirements." https://www.ncaa.org/eligibility-center/initial-eligibility-requirements/division-ii/. Accessed 30 August 2026. 16 core courses including 2 years mathematics (Algebra I or higher); 2.2 core GPA. [Cited]

NCAA Eligibility Center. *NCAA Home School Toolkit 2025–26*. http://fs.ncaa.org/Docs/eligibility_center/ Accessed 30 August 2026. Transcript elements; core-course definition; "Audited, CLEP or credit-by-exam courses are not NCAA-approved core courses." [Cited]

NCAA Eligibility Center. *Homeschool Transcript Information / Example*. http://fs.ncaa.org/Docs/eligibility_center/Student_Resources/Home_School_Transcript_Example.pdf. Accessed 30 August 2026. Example showing Algebra I, Geometry, Algebra II, Precalculus sequence and required fields. [Cited]

National Center for Education Statistics. *Dual or Concurrent Enrollment in Public Schools in the United States*. NCES 2020-125. Washington, DC: NCES, December 2020. https://nces.ed.gov/pubs2020/2020125.pdf. Accessed 30 August 2026. 2017–18 NTPS: 82% of public schools with grades 9–12 offered dual enrollment. Public-school *supply*, not homeschool participation. [Cited]

National Council of Teachers of Mathematics (Jane Schielack et al.). *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*. 2006. Grade-by-grade excerpt posting, https://www.platonicsolids.info/focal_points_by_grade.pdf. Accessed 30 August 2026. Third-party posting of NCTM 2006 grade text. Secondary copy of a primary; confirm against a library copy before quotation. Official book paywalled. [Cited; UNVERIFIED against official volume]

National Council of Teachers of Mathematics. "Curriculum Focal Points" landing page. 2006. https://www.nctm.org/curriculumfocalpoints/. Accessed 30 August 2026. Official product/landing page. Full book paywalled. [Consulted]

National Council of Teachers of Mathematics. *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000. https://www.nctm.org/Standards-and-Positions/Principles-and-Standards/. Accessed 30 August 2026. Five content standards and five process standards (Problem Solving; Reasoning and Proof; Communication; Connections; Representation — that order) across four grade bands. Full 402-page book paywalled. Process order corrected from the 2000 primary (front-errata). [Cited]

National Council of Teachers of Mathematics. *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: NCTM, 2014. https://www.nctm.org/PtA/. Accessed 30 August 2026. Page 42: "procedural fluency follows and builds on a foundation of conceptual understanding." Policy stance, not an experiment tested against a procedures-then-concepts alternative (Rittle-Johnson, Schneider, and Star 2015). [Cited]

National Council of Teachers of Mathematics. *Principles and Standards for School Mathematics: Executive Summary*. 2000. https://www.nctm.org/uploadedFiles/Standards_and_Po
Accessed 30 August 2026. Official executive summary. [Cited]

National Council of Teachers of Mathematics. "What is Principles and Standards for School Mathematics?" FAQ. 2000. https://www.nctm.org/uploadedFiles/Standards_and_Positions/Pri
Accessed 30 August 2026. 10 standards, four grade bands, content versus process. [Cited]

National Mathematics Advisory Panel. *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education, 2008. ERIC ED500486, https://files.eric.ed.gov/fulltext/ED500486.pdf. Accessed 30 August 2026. Finding 4 / Chapter 4 fractions gate; Finding 10 capabilities "mutually supportive" (not "mutually reinforcing"); Finding 15 two sentences kept separate (developmental appropriateness "largely contingent on prior opportunities to learn"; age-lock claims "consistently been shown to be wrong"); Finding 26 kitchen-math; Finding 27 explicit instruction for students already struggling. Table 2 geometry/measurement remainder truncated in the ERIC fetch. Memo-locked fractions sentence left standing; exact "among American students" string not relocated in the final-report PDF this pass. [Cited]

National Mathematics Advisory Panel Task Group. Conceptual Knowledge and Skills Task Group report excerpts (Major Topics of School Algebra). 2008.

http://www.ed.gov/media/document/conceptual-knowledgepdf-89273.pdf. Accessed 30 August 2026. ED-hosted task-group material listing Major Topics of School Algebra. Prefer the Final Report Table 1 for quotation. [Consulted]

National Research Council. *Adding It Up: Helping Children Learn Mathematics*. Edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell. Washington, DC: National Academy Press, 2001. https://nap.nationalacademies.org/catalog/9822/adding-it-up-helping-children-learn-mathematics. Also https://doi.org/10.17226/9822. Five strands p. 116, “interwoven and interdependent.” Extreme-positions sentence is p. 409 (ch. 11), not p. 116 (front-errata). NMAP footnote 2 cites p. 116 for the five attributes, not the extreme-positions sentence. [Cited]

National Research Council. *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*. 2009. https://nap.nationalacademies.org/catalog/12519/mathematics-learning-in-early-childhood-paths-toward-excellence-and-equity. Cited on the first page of the CCSSM PDF (number and geometry as early foci). Full book not re-opened this pass. [Consulted; UNVERIFIED full text this pass]

Ohio Department of Education. *K-8 Standards Progressions*. 2012. State restatement of CCSSM domains across grades. https://cdnsm5-ss18.sharpschool.com/UserFiles/Servers/Server_
8%20Standards%20Progressions%202%2014%2012.pdf. Useful domain map; not an independent standard. [Consulted]

UK Department for Education. “National curriculum in England: mathematics programmes of study.” 2013. https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study/national-curriculum-in-england-mathematics-programmes-of-study. Accessed 30 August 2026. Crown copyright / OGL. Year-by-year KS1–2 and KS3–4 including KS3 algebra (linear and simple quadratic graphs). Opened in full 30 August 2026. [Cited]

---

2. Empirical learning science

American Psychiatric Association. *Diagnostic and Statistical Manual of Mental Disorders*. 5th ed. Arlington, VA: American Psychiatric Publishing, 2013. Specific learning disorder (mathematics). Public page, https://www.psychiatry.org/patients-families/specific-learning-disorder/what-is-specific-learning-disorder. Accessed 30 August 2026. 5–15% is all SLDs, not dyscalculia. Criteria paraphrased from the APA public page and Butterworth 2019 preview. [Cited]

Ashcraft, Mark H., and Elizabeth P. Kirk. "The Relationships among Working Memory, Math Anxiety, and Performance." *Journal of Experimental Psychology: General* 130, no. 2 (2001): 224–37. Computation-span $r = -.40$, Experiments 1 and 3. Full 2001 PDF not opened this pass; coefficient from Ashcraft's later quotation of those experiments. [Cited; UNVERIFIED full PDF]

Ashcraft, Mark H., and Jeremy A. Krause. Later reviews (2007 and after), often cited for middle-school onset as mathematics becomes more working-memory intensive. Not re-opened this pass. Full titles/pages omitted rather than guessed. [Cited; UNVERIFIED full texts]

Assouline, Susan G., Nicholas Colangelo, Joyce VanTassel-Baska, and colleagues. Nation Empowered / Iowa Acceleration Scale literature. Not opened this pass; effect sizes UNVERIFIED. Remaining authors and full citation not locked. Do not invent a protocol or a prevalence. [Cited; UNVERIFIED]

Barnett, Susan M., and Stephen J. Ceci. Transfer-taxonomy paper (2002). Named in manuscript notes; paper not opened; exact dimensions list UNVERIFIED. Full title/pages omitted rather than guessed. [Cited; UNVERIFIED]

Baroody, Arthur J., and Herbert P. Ginsburg. "The Effects of Instruction on Children's Understanding of the Equals Sign." *Elementary School Journal* 83 (1983). https://doi.org/10.1086/461356. Cited from McNeil and Booth. Full PDF not opened. [Cited; UNVERIFIED full text]

Beilock, Sian L., Elizabeth A. Gunderson, Gerardo Ramirez, and Susan C. Levine. "Female Teachers' Math Anxiety Affects Girls' Math Achievement." *Proceedings of the National Academy of Sciences* (2010). https://doi.org/10.1073/pnas.0910967107. Fetched at secondary level; coefficients UNVERIFIED this pass. Not quoted as if read. [Cited; UNVERIFIED coefficients]

Booth, Julie L., Karin E. Lange, Kenneth R. Koedinger, and Kristie J. Newton. "Using Example Problems to Improve Student Learning in Algebra: Differentiating between Correct and Incorrect Examples." *Learning and Instruction* 25 (2013): 24–34. ERIC ED543090, https://files.eric.ed.gov/fulltext/ED543090.pdf. Accessed 30 August 2026. Second author is Karin E. Lange, not Kenneth (mid-2 errata). Incorrect-example benefit for conceptual understanding taken from the paper. [Cited]

Butterworth, Brian. *Dyscalculia: From Science to Education*. 2019. https://doi.org/10.4324/9781315538136. Preview pages fetched for DSM-5 wording critique. Research program, not a prevalence census. [Cited; UNVERIFIED full volume]

Canobi, Katherine H. “Concept-Procedure Interactions in Children’s Addition and Subtraction.” *Journal of Experimental Child Psychology* 102 (2009): 131–49. https://doi.org/10.1016/j.jecp.2008.07.008. Cited from Rittle-Johnson, Schneider, and Star 2015. Conceptually sequenced practice beat random order. [Cited]

Carbonneau, Kira J., Scott C. Marley, and James P. Selig. “A Meta-Analysis of the Efficacy of Teaching Mathematics with Concrete Manipulatives.” *Journal of Educational Psychology* 105, no. 2 (2013): 380–400. https://doi.org/10.1037/a0031084. Overall mean *d* approximately 0.37 versus abstract-only (55 studies, N = 7,237), labelled approximate: full PDF not opened; figure from abstract plus quoting sources. Larger on retention than transfer or justification. [Cited; UNVERIFIED full PDF]

Cepeda, Nicholas J., Harold Pashler, Edward Vul, John T. Wixted, and Doug Rohrer. “Distributed Practice in Verbal Recall Tasks: A Review and Quantitative Synthesis.” *Psychological Bulletin* 132 (2006): 354–80. https://doi.org/10.1037/0033-2909.132.3.354. Cited from Rohrer 2014 and Dunlosky 2013. Full PDF not opened. No mashed effect size used. [Cited; UNVERIFIED full text]

Confrey, Jere, Alan P. Maloney, and Kenny H. Nguyen. “Breaching the Conditions for Success for a National Advisory Panel.” *Educational Researcher* (2008). https://doi.org/10.3102/0013189X08329194. Methodological critique of NMAP 2008. Abstract fetched. Report alongside the Panel. [Cited]

Duncan, Greg J., and colleagues. “School Readiness and Later Achievement.” *Developmental Psychology* 43 (2007): 1428–46. https://doi.org/10.1037/0012-1649.43.6.1428. Cited from Siegler et al. 2012. Early math more stable than reading. Full PDF not opened. Remaining authors after Duncan not locked from a title-page refetch this pass. [Cited; UNVERIFIED full text]

Dunlosky, John, Katherine A. Rawson, Elizabeth J. Marsh, Mitchell J. Nathan, and Daniel T. Willingham. “Improving Students’ Learning with Effective Learning Techniques: Promising Directions from Cognitive and Educational

Psychology." *Psychological Science in the Public Interest* 14, no. 1 (2013): 4–58. https://doi.org/10.1177/1529100612453266. Practice testing and distributed practice rated high-utility; interleaved practice and self-explanation moderate-utility. Do not mash with Rohrer classroom *d* values. [Cited]

Frye, Douglas, Arthur J. Baroody, Margaret Burchinal, Sharon M. Carver, Nancy C. Jordan, and Judy McDowell. *Teaching Math to Young Children*. NCEE 2014-4005. Washington, DC: Institute of Education Sciences, 2013. https://ies.ed.gov/ncee/wwc/PracticeGuide/18. Accessed 30 August 2026. Ages 3–6; five recommendations from the official page. [Cited]

Fuchs, Lynn S., and colleagues. "Improving At-Risk Learners' Understanding of Fractions." *Journal of Educational Psychology* (2013). Author PDF located, https://siegler.tc.columbia.edu/wp-content/uploads/2019/08/5023-Fuchs13.pdf. Accessed 30 August 2026. Numerical effect sizes not copied into the memos. Do not invent a *g*. Remaining co-authors not locked from a title-page refetch this pass. [Cited; UNVERIFIED coefficients]

Fuchs, Lynn S., Rebecca Newman-Gonchar, Robin Schumacher, Barbara Dougherty, Nicole Bucka, Karen S. Karp, John Woodward, et al. *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*. WWC 2021006. Washington, DC: Institute of Education Sciences, 2021. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/WWC2021006-Math-PG.pdf. Accessed 30 August 2026. Full PDF fetched 30 August 2026. Recommendations 1–6 study counts 43 / 16 / 28 / 14 / 18 / 27. Chicago 17: first seven of thirteen authors, then et al. [Cited]

Fuson, Karen C., and Diane J. Briars. "Using a Base-Ten Blocks Learning/Teaching Approach for First- and Second-Grade Place-Value and Multidigit Addition and Subtraction." *Journal for Research in Mathematics Education* 21, no. 3 (1990): 180–206. https://doi.org/10.2307/749373. Cited from Rittle-Johnson, Schneider, and Star 2015. Pages as given in manuscript notes. Full PDF not opened. [Cited; UNVERIFIED full text]

Gelman, Rochel, and C. R. Gallistel. *The Child's Understanding of Number*. Cambridge, MA: Harvard University Press, 1978. https://www.hup.harvard.edu/books/9780674116375. Five counting principles from later papers that quote the 1978 text (Sarnecka and Carey 2008; Stanford counting-principles brief). Book itself not re-opened. [Cited; UNVERIFIED full volume this pass]

Gersten, Russell, Sybilla Beckmann, Ben Clarke, Anne Foegen, Laurel Marsh, Jon R. Star, and Bradley Witzel. *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools*. NCEE 2009-4060. Washington, DC: Institute of Education Sciences, 2009. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/rti_math_pg_042109.pdf. Accessed 30 August 2026. About 10 minutes per intervention session on fact fluency; rejects keyword strategies. Full long PDF not re-read cover-to-cover. [Cited]

Hecht, Steven A., and Kevin J. Vagi. "Sources of Group and Individual Differences in Emerging Fraction Skills." *Journal of Educational Psychology* 102 (2010): 843–58. https://doi.org/10.1037/a0019824. Cited from Rittle-Johnson 2015 and Siegler 2012. Bidirectional fraction knowledge over grades 4–5. [Cited]

Hiebert, James, and Douglas A. Grouws. "The Effects of Classroom Mathematics Teaching on Students' Learning." In *Second Handbook of Research on Mathematics Teaching and Learning*, edited by Frank K. Lester. Charlotte, NC: Information Age / NCTM, 2007. Productive-struggle construct from Kapur 2014 and Rittle-Johnson 2015 citations. Full chapter not opened. [Cited; UNVERIFIED full chapter]

Hyde, Janet S., and colleagues. 2006 paper on quality of mothers' scaffolding on fifth-grade mathematics homework, as cited in later work. Named reminder that help is not one behavior; not re-opened as a primary this pass. Remaining authors and full citation not locked. [Cited; UNVERIFIED full text]

Jitendra, Asha K., Jon R. Star, Danielle Dupuis, and Michael C. Rodriguez. "Effectiveness of Schema-Based Instruction for Improving Seventh-Grade Students' Proportional Reasoning: A Randomized Experiment." *Journal of Research on Educational Effectiveness* 5 (2012). https://doi.org/10.1080/19345747.2012.725804. Immediate g = 1.24; six-week retention g = 1.27; null transfer; N = 1,163, 42 classrooms. Do not mash with Rohrer 2014. [Cited]

Jitendra, Asha K., and colleagues. "A Randomized Controlled Trial of the Impact of Schema-Based Instruction on Mathematical Outcomes for Third-Grade Students with Mathematics Difficulties." *Elementary School Journal* 114, no. 2 (2013): 252–76. https://doi.org/10.1086/673199. Word-problem g = 0.46; district g = 0.34; null 8-week retention. Remaining co-authors not locked from a title-page refetch this pass. [Cited]

Kalyuga, Slava, Paul Ayres, Paul Chandler, and John Sweller. "The Expertise Reversal Effect." *Educational Psychologist* 38, no. 1 (2003): 23–31.

https://doi.org/10.1207/S15326985EP3801_4. Cited from Sweller 2019. Full PDF not opened. [Cited; UNVERIFIED full text]

Kapur, Manu. “Productive Failure in Learning Math.” *Cognitive Science* 38, no. 5 (2014): 1008–22. https://doi.org/10.1111/cogs.12107. Two randomized studies; PF greater conceptual and transfer, comparable procedural. Different comparison from NMAP Finding 27; do not use to cancel Finding 27. [Cited]

Kieran, Carolyn. “Concepts Associated with the Equality Symbol.” *Educational Studies in Mathematics* (1981). https://doi.org/10.1007/BF00311062. Classic equal-sign paper. Cited from McNeil and Booth. Full PDF not opened. [Cited; UNVERIFIED full text]

Kirschner, Paul A., John Sweller, and Richard E. Clark. “Why Minimal Guidance During Instruction Does Not Work.” *Educational Psychologist* 41, no. 2 (2006): 75–86. https://doi.org/10.1207/s15326985ep4102_1. Position/review paper, not a new experiment. [Cited]

Knuth, Eric J., Ana C. Stephens, Nicole M. McNeil, and Martha W. Alibali. “Does Understanding the Equal Sign Matter? Evidence from Solving Equations.” *Journal for Research in Mathematics Education* 37, no. 4 (2006): 297–312. https://doi.org/10.2307/30034852. Cited from Booth 2013 and McNeil reviews. Full PDF not opened. [Cited; UNVERIFIED full text]

Ma, Liping. *Knowing and Teaching Elementary Mathematics*. Mahwah, NJ: Erlbaum, 1999. https://doi.org/10.4324/9781410603784. U.S. versus Chinese teacher knowledge of invert-and-multiply, as cited by Siegler et al. 2012. Book not re-opened. [Cited; UNVERIFIED full volume this pass]

Maloney, Erin A., Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock. “Intergenerational Effects of Parents’ Math Anxiety on Children’s Math Achievement and Anxiety.” *Psychological Science* 26, no. 9 (2015): 1480–88. https://doi.org/10.1177/0956797615592630. Grades 1–2; effect only when math-anxious parents reported frequent homework help; math-specific, not reading. School-homework study, not a parent-as-sole-teacher RCT. Single large field study; replication status open. [Cited]

McNeil, Nicole M., Emily R. Fyfe, Lori A. Petersen, April E. Dunwiddie, and Heather Brletic-Shipley. “Benefits of Practicing 4 = 2 + 2: Nontraditional Problem Formats Facilitate Children’s Understanding of Mathematical Equivalence.” *Child Development* 82 (2011): 1620–33. https://doi.org/10.1111/j.1467-

8624.2011.01635.x. Cited from Rittle-Johnson 2015. [Cited]

McNeil, Nicole M. “A Change-Resistance Account of Children’s Difficulties Understanding Mathematical Equivalence.” *Child Development Perspectives* (2014). Lab PDF, https://cladlab.nd.edu/assets/258752/mcneil_2014_child_development_perspectives.pdf. Accessed 30 August 2026. [Cited]

McNeil, Nicole M., and Martha W. Alibali. “Why Won’t You Change Your Mind? Knowledge of Operational Patterns Hinders Learning and Performance on Equations.” *Child Development* 76, no. 4 (2005): 883–99. Lab PDF, https://cladlab.nd.edu/assets/250421/mcneilalibali05b.pdf. Accessed 30 August 2026. Three operational patterns; negative relation to equation learning. [Cited]

Morsanyi, Kinga, Jemma H. C. van Bers, Teresa McCormack, and Juhani McGourty. “The Prevalence of Specific Learning Disorder in Mathematics and Comorbidity with Other Developmental Disorders in Primary School-Age Children.” *British Journal of Psychology* lineage PDF. 2018. https://pureadmin.qub.ac.uk/ws/portalfiles/portal/154631447/The_prevalence_of_developmental_dyscalc One sample of 2,421; 6% persistent SLDM under DSM-5 cuts. Not a universal prevalence. [Cited]

Ni, Yujing, and Yong-Di Zhou. “Teaching and Learning Fraction and Rational Numbers: The Origins and Implications of Whole Number Bias.” *Educational Psychologist* 40, no. 1 (2005): 27–52. https://doi.org/10.1207/s15326985ep4001_3. Three origin accounts; instructional implications from the abstract. [Cited]

Peltier, Corey, and colleagues. “We’re Talking about Practice: A Meta-Analysis of Timed Math Practice.” OSF preprint (EdArXiv), 2025. https://doi.org/10.35542/osf.io/9bjtg. Labelled preprint. Argues IES 2021 timed-activities studies do not isolate the timer. Remaining authors not locked. [Cited; Preprint]

Peltier, Corey, and Kimberly J. Vannest. “A Meta-Analysis of Schema Instruction on the Problem-Solving Performance of Elementary School Students.” *Review of Educational Research* (2017). https://doi.org/10.3102/0034654317720163. Hedge’s g = 1.57 immediate problem solving, g = 1.09 transfer; 21 studies; 3,408 students. Mostly proximal measures; not a general-achievement effect. [Cited]

Renkl, Alexander, and Robert K. Atkinson. “Structuring the Transition from Example Study to Problem Solving in Cognitive Skill Acquisition: A Cognitive Load Perspective.” *Educational Psychologist* 38 (2003): 15–22. https://doi.org/10.1207/S15326985EP3801_3. Guidance-fading / backward fading. Cited from Sweller 2019. Full PDF not opened. [Cited; UNVERIFIED full text]

Retanal, Fraulein, Noah Johnston, Sabrina M. Di Lonardo Burr, Andie Storozuk, Michela DiStefano, and Erin A. Maloney. "Controlling-Supportive Homework Help Partially Explains the Relation between Parents' Math Anxiety and Children's Math Achievement." *Education Sciences* 11, no. 10 (2021): 620. https://doi.org/10.3390/educsci11100620. Survey of parents of ages 11–14; path analysis; partial mediator. Not a kitchen RCT. [Cited]

Rittle-Johnson, Bethany. "Promoting Transfer: Effects of Self-Explanation and Direct Instruction." *Child Development* 77 (2006): 1–15. https://doi.org/10.1111/j.1467-8624.2006.00852.x. Cited from Rittle-Johnson 2015. Full PDF not opened. [Cited; UNVERIFIED full text]

Rittle-Johnson, Bethany, and Martha W. Alibali. "Conceptual and Procedural Knowledge of Mathematics: Does One Lead to the Other?" *Journal of Educational Psychology* 91, no. 1 (1999): 175–89. https://doi.org/10.1037/0022-0663.91.1.175. Cited and summarized in the 2015 review. Full PDF not opened. [Cited; UNVERIFIED full text]

Rittle-Johnson, Bethany, and Kenneth R. Koedinger. "Iterating between Lessons on Concepts and Procedures Can Improve Mathematics Knowledge." *British Journal of Educational Psychology* 79 (2009): 483–500. https://doi.org/10.1348/000709908X398106. Two classroom experiments, sixth-grade decimals, N = 77 and 26. Favored iteration, not a long conceptual foundation. [Cited]

Rittle-Johnson, Bethany, Michael Schneider, and Jon R. Star. "Not a One-Way Street: Bidirectional Relations Between Procedural and Conceptual Knowledge of Mathematics." *Educational Psychology Review* 27, no. 4 (2015): 587–97. Full PDF, https://www.uni-trier.de/fileadmin/fb1/prof/PSY/PAE/Team/Schneider/Rittle-JohnsonEtAl2015.pdf. Accessed 30 August 2026. "It is a myth that it is a one-way street from conceptual knowledge to procedural knowledge." [Cited]

Rittle-Johnson, Bethany, Robert S. Siegler, and Martha W. Alibali. "Developing Conceptual Understanding and Procedural Skill in Mathematics: An Iterative Process." *Journal of Educational Psychology* 93, no. 2 (2001): 346–62. https://doi.org/10.1037/0022-0663.93.2.346. Full PDF paywalled. Coefficients described qualitatively in the paper's abstract. [Cited; UNVERIFIED full PDF]

Rohrer, Doug, Robert F. Dedrick, and Kaleena Burgess. "The Benefit of Interleaved Mathematics Practice Is Not Limited to Superficially Similar Kinds of Problems." *Psychonomic Bulletin & Review* 21 (2014): 1323–30. ERIC ED548041, https://files.eric.ed.gov/fulltext/ED548041.pdf. Accessed 30 August 2026. Grade 7; n = 140; unannounced test two weeks later; interleaved 72% versus blocked 38%; d = 1.05, 95% CI [0.80, 1.30]. Do not mash with Rohrer 2020 or with Dunlosky "high utility." [Cited]

Rohrer, Doug, Robert F. Dedrick, Marissa K. Hartwig, and Chi-Ngai Cheung. "A Randomized Controlled Trial of Interleaved Mathematics Practice." *Journal of Educational Psychology* 112, no. 1 (2020): 40–52. https://doi.org/10.1037/edu0000367. Fourth author is Chi-Ngai Cheung, not Alexander (mid-2 errata). Preregistered cluster RCT, 54 seventh-grade classes; interleaved 61% versus blocked 38%; d = 0.83; one-month delay; joint with spacing. [Cited]

Rohrer, Doug, Robert F. Dedrick, and Marissa K. Hartwig. "The Scarcity of Interleaved Practice in Mathematics Textbooks." *Educational Psychology Review* 32 (2020): 873–83. https://doi.org/10.1007/s10648-020-09516-2. Textbooks almost entirely blocked. [Cited]

Rohrer, Doug, and Kelli Taylor. "The Effects of Overlearning and Distributed Practice on the Retention of Mathematics Knowledge." *Applied Cognitive Psychology* 20 (2006): 1209–24. https://doi.org/10.1002/acp.1266. Cited from Rohrer 2014. Full PDF not opened. [Cited; UNVERIFIED full text]

Rohrer, Doug, and Kelli Taylor. "The Shuffling of Mathematics Practice Problems Boosts Learning." *Instructional Science* 35 (2007): 481–98. https://doi.org/10.1007/s11251-007-9015-8. Cited from Rohrer 2014. Full PDF not opened. [Cited; UNVERIFIED full text]

Schneider, Michael, Bethany Rittle-Johnson, and Jon R. Star. "Relations between Conceptual Knowledge, Procedural Knowledge, and Procedural Flexibility." *Developmental Psychology* 47, no. 6 (2011): 1525–38. https://doi.org/10.1037/a0024997. Cited from the 2015 review. Full PDF not opened. Pages as given in manuscript notes. [Cited; UNVERIFIED full text]

Siegler, Robert, Thomas Carpenter, Francis Fennell, David Geary, James Lewis, Yukari Okamoto, Laurie Thompson, and Jonathan Wray. *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*. NCEE 2010-4039. Washington, DC: Institute of Education Sciences, 2010.

https://ies.ed.gov/ncee/wwc/docs/practiceguide/fractions_pg_093010.pdf. Accessed 30 August 2026. Recommendation 2: missing magnitude “often at the root” of adding numerators and denominators. [Cited]

Siegler, Robert S., Greg J. Duncan, Pamela E. Davis-Kean, Kathryn Duckworth, Amy Claessens, Mimi Engel, Maria Ines Susperreguy, and Meichu Chen. “Early Predictors of High School Mathematics Achievement.” *Psychological Science* 23, no. 7 (2012): 691–97. Author PDF, https://siegler.tc.columbia.edu/wp-content/uploads/2019/02/Siegler-etal-PsySci12.pdf. Also https://doi.org/10.1177/0956797612440101. Accessed 30 August 2026. Seventh author as published: Maria Ines Susperreguy (accents not in the primary; front-errata). British Cohort Study N = 3,677; U.S. PSID-CDS N = 599. 1-SD age-10 fractions uniquely associated with 0.15-SD (UK) / 0.17-SD (US) high-school algebra and 0.16 / 0.18 overall high-school math after controls; concurrent r = .81 / .87. Prediction, not an RCT. [Cited]

Siegler, Robert S., and Hugues Lortie-Forgues. “Conceptual Knowledge of Fraction Arithmetic.” 2015. Author PDF, https://siegler.tc.columbia.edu/wp-content/uploads/2019/02/2015-Siegler-Lortie-Forgues.pdf. Accessed 30 August 2026. Documents whole-number bias in fraction arithmetic. [Cited]

Siegler, Robert S., Clarissa A. Thompson, and Michael Schneider. “An Integrated Theory of Whole Number and Fractions Development.” *Cognitive Psychology* 62, no. 4 (2011): 273–96. https://doi.org/10.1016/j.cogpsych.2011.03.001. Cited and quoted via Siegler 2012. Full 2011 PDF not separately opened. [Cited]

Soderstrom, Nicholas C., and Robert A. Bjork. “Learning Versus Performance: An Integrative Review.” *Perspectives on Psychological Science* 10, no. 2 (2015): 176–99. https://doi.org/10.1177/1745691615569000. Author PDF, https://bjorklab.psych.ucla.edu/wp-content/uploads/sites/13/2016/11/soderstorm_ra_learningvsperformar Accessed 30 August 2026. Current performance is “often an unreliable index” of learning (p. 176). [Cited]

Star, Jon R. “Reconceptualizing Procedural Knowledge.” *Journal for Research in Mathematics Education* 36 (2005): 404–11. https://doi.org/10.2307/30034943. Distinguishes knowledge type versus quality. Cited from Rittle-Johnson 2015. [Cited]

Stockard, Jean, Timothy W. Wood, Cristy Coughlin, and Caitlin Rasplica Khoury. “The Effectiveness of Direct Instruction Curricula: A Meta-Analysis of a Half Century of Research.” *Review of Educational Research* 88, no. 4 (2018): 479–

507. https://doi.org/10.3102/0034654317751919. Total-sample effect 0.60, 95% CI [0.54, 0.66]; 328 studies. Named DI curriculum family, not "lecture." Authors associated with DI. [Cited]

Sweller, John. "Cognitive Load During Problem Solving: Effects on Learning." *Cognitive Science* 12 (1988): 257–85. https://doi.org/10.1207/s15516709cog1202_4. Foundational CLT paper. Full PDF not opened. [Cited; UNVERIFIED full text]

Sweller, John, Jeroen J. G. van Merriënboer, and Fred Paas. "Cognitive Architecture and Instructional Design: 20 Years Later." *Educational Psychology Review* 31 (2019): 261–92. https://doi.org/10.1007/s10648-019-09465-5. Guidance-fading and expertise-reversal summarized from the Springer page. [Cited]

Sweller, John, and Graham A. Cooper. "The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra." *Cognition and Instruction* 2, no. 1 (1985): 59–89. https://doi.org/10.1207/s1532690xci0201_3. Foundational worked-example paper. Abstract/secondary fetched; full PDF not opened. No invented *d*. [Cited; UNVERIFIED full text]

Tall, David, and Shlomo Vinner. "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity." *Educational Studies in Mathematics* 12 (1981): 151–69. Named in manuscript notes; paper not opened this project. Do not invent a teaching sequence or quote a concept-image sentence as if the PDF had been read. [Cited; UNVERIFIED full text]

Taylor, Kelli, and Doug Rohrer. "The Effects of Interleaving Practice." *Applied Cognitive Psychology* 24 (2010): 837–48. https://doi.org/10.1002/acp.1598. Both conditions spaced; d = 1.23; blocked errors mostly wrong-strategy. Cited from Rohrer 2014; full PDF not opened. [Cited; UNVERIFIED full text]

Vosniadou, Stella, Xenia Vamvakoussi, and Irini Skopeliti. "The Framework Theory Approach to Conceptual Change." In *International Handbook of Research on Conceptual Change*, edited by Stella Vosniadou. 2008. https://doi.org/10.4324/9780203802175. Cited from Siegler 2012 for persistent whole-number/fraction confusion. Full chapter not opened. [Cited; UNVERIFIED full chapter]

Wai, Jonathan, David Lubinski, and Camilla P. Benbow. 2009 spatial-skill and later STEM achievement paper. Named in manuscript notes; paper not opened; exact coefficients UNVERIFIED. Full title/pages omitted rather than guessed. [Cited; UNVERIFIED]

Woodward, John, and colleagues. *Improving Mathematical Problem Solving in Grades 4 Through 8*. NCEE 2012-4055. Washington, DC: Institute of Education Sciences, 2012. https://ies.ed.gov/ncee/wwc/PracticeGuide/16. Existence and IES URL confirmed. Full text UNVERIFIED this pass. Remaining authors not locked. [Consulted; UNVERIFIED full text]

---

3. Homeschool context, law, and outcomes

U.S. Census Bureau. Household Pulse Survey Phase 4.2 Cycle 09 Education Tables (20 August–16 September 2024). https://www.census.gov/data/tables/2024/demo/hhp/cycle09.html. Accessed 30 August 2026. Federal experimental survey. NCES warns against comparing Pulse to NHES. Cited by NHERI for a 6.530% figure. [Cited]

U.S. Department of Education. 34 C.F.R. §§ 300.130–300.144 (equitable services for parentally placed private school children with disabilities). eCFR, https://www.ecfr.gov/current/title-34/subtitle-B/chapter-III/part-300/subpart-B/subject-group-ECFR3556f7ac2fe0a92/section-300.137. Accessed 30 August 2026. Current as of 27 August 2026 (title last amended 24 July 2026). § 300.137(a): no individual right to FAPE for parentally placed private-school children. [Cited]

Cheng, Albert, and Angela Watson. *Diverse Outcomes for a Diverse Population: Findings About Homeschooled Adults from the Cardus Education Survey*. Cardus, 2024. https://www.cardus.ca/research/diverse-outcomes-for-a-diverse-population/. Accessed 30 August 2026. CES fielded 10–22 October 2023; Ipsos KnowledgePanel; 2,350 U.S. adults ages 24–39; n = 181 ever-homeschooled. Bachelor's: medium-term 29%, long-term 27%, never-homeschooled 46%, after demographic controls. Only 17% of ever-homeschooled adults were homeschooled all K–12. Think tank, not a federal agency. Authors refuse a causal interpretation. Does not report math scores. [Cited]

Common App Member Support. "How can I create a recommender account?" 7 May 2024. https://membersupport.commonapp.org/s/article/I-am-a-counselor-or-recommender-How-do-I-create-my-account. Accessed 30 August 2026. Counselor account is invitation-only after the student assigns a recommender — the mechanism a homeschool parent uses as counselor. Current counselor-form homeschool questions not reprinted in the text opened. [Cited]

Cui, Jiashan, and Rachel Hanson. *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey*. NCES 2020-001. Washington, DC: National Center for Education Statistics, 2019. https://nces.ed.gov/pubs2020/2020001.pdf. Accessed 30 August 2026. Web Tables, not the 2016 First Look (mid-2 errata). Table 3: mother 78% main provider; tutor 23%; co-op 31%. Table 6: 77% mostly or strictly formal; 12% mostly or strictly informal. Table 8: ever-taught Algebra I 79%, geometry 52%, Algebra II 49%, calculus 15% (caution). Table 9: Algebra I *that year* 41% of 6–8; arithmetic 31% of 9–12. The 2023 First Look did not republish these tables. [Cited]

Dee, Thomas. *Where the Kids Went: Nonpublic Schooling and Demographic Change During the Pandemic Exodus from Public Schools*. Urban Institute, 2023. https://www.urban.org/research/publication/where-kids-went-nonpublic-schooling-and-demographic-change-during-pandemic. Accessed 30 August 2026. Public-school enrollment decline and homeschool increase 2019–2021. Not a math-outcome paper. [Cited]

Hamlin, Daniel, and Albert Cheng. "Homeschooling, Perceived Social Isolation, and Life Trajectories: An Analysis of Formerly Homeschooled Adults." *Journal of School Choice* (2022). https://doi.org/10.1080/15582159.2022.2028338. Duration of homeschooling as a heterogeneity dimension; cited by Cardus 2024 for the short/medium/long split. Paywalled DOI. [Cited]

Kunzman, Robert, and Milton Gaither. "Homeschooling: An Updated Comprehensive Survey of the Research." *Other Education* 9, no. 1 (2020): 253–336. https://icher.org/files/Kunzman_and_Gaither_An%20Updated_Comprehensive_Survey.pdf. Accessed 30 August 2026. Review of 2,000+ English-language texts. Treats Ray/NHERI as advocacy-era research. Notes curriculum research is thin relative to demographics and law. [Cited]

Lubienski, Christopher, Tiffany Puckett, and T. Jameson Brewer. "Does Homeschooling 'Work'? A Critique of the Empirical Claims and Agenda of Advocacy Organizations." *Peabody Journal of Education* 88, no. 3 (2013). https://doi.org/10.1080/0161956X.2013.798516. Peer-reviewed critique. Selection effects; advocacy claims. Paywalled DOI. [Cited]

Sempeles, Ellena, and Jiashan Cui. *Parent and Family Involvement in Education: 2023*. NCES 2024-113. Washington, DC: National Center for Education Statistics, September 2024. https://nces.ed.gov/pubs2024/2024113.pdf. Accessed

30 August 2026. NHES:2023 PFI. Table A-6: 3.4% homeschooled (~1,765,000 students ages 5–17 with a K–12 grade equivalent, 2022–23); 5.2% instruction at home. Weighted *n* = 19,562 representing ~53 million K–12. Latest federal PFI opened; **not a 2026 national total**. [Cited]

Sempeles, Ellena, and Jiashan Cui. *Parent and Family Involvement in Education: 2023* (First Look—Summary). NCES 2024-113. 2024. https://nces.ed.gov/pubs2024/2024113_Summary. Accessed 30 August 2026. Same estimates as the full First Look. [Cited]

National Center for Education Statistics. "Homeschooled Children and Reasons for Homeschooling." *Condition of Education*, May 2022. https://nces.ed.gov/programs/coe/indicator/tgk/ children. Accessed 30 August 2026. NHES 1999–2019 trend (2.8% / 1,457,000 in 2019) plus experimental Household Pulse Survey 2020–21 (5.4% child-level). NCES: Pulse is not comparable to NHES (6.6% response in the cited wave). [Cited]

National Center for Education Statistics / IES. "A Higher Percentage of K–12 Students are Receiving Academic Instruction at Home." Press release, 17 September 2024. https://ies.ed.gov/learn/press-release/higher-percentage-k-12-students-are-receiving-academic-instruction-home. Accessed 30 August 2026. Agency voice. Same 5.2% / 3.4% figures. [Cited]

National Center for Systemic Improvement (WestEd). *Quick Reference Guide on Parentally-Placed Private School Proportionate Share of IDEA Funds*. 2025. https://ncsi.wested.org/wp-content/uploads/2025/01/NCSI_Quick_Reference_Proportionate_Share_FINA Accessed 30 August 2026. If a state recognizes homeschools as private schools, those children are in the proportionate-share count. Not binding law. [Cited]

North Central College Admissions. "Homeschool Applicants." https://northcentralcollege.edu/apply/hom applicants. Accessed 30 August 2026. Recommends 3 years mathematics including algebra and geometry, one course beyond geometry preferred. Named page, not NACAC policy. [Cited]

NSW Department of Education. "Curriculum Reform and home schooling." https://www.nsw.gov.au/education-and-training/home-schooling/support-material/curriculum-reform-home-schooling. Accessed 30 August 2026. Home education programs must be based on NESA Mathematics K–10 syllabuses on stated timelines. [Cited]

NSW Department of Education. "Home schooling registration at a glance." https://www.nsw.gov.au/education-and-training/home-schooling/registration-guidelines/registration-at-a-glance. Accessed 30 August 2026. Australian state government primary. Registered programs must include Mathematics; Authorised Person assesses the program, not the child. [Cited]

U.S. Department of Education, OSERS/OSEP. *Questions and Answers on Serving Children with Disabilities Placed by Their Parents in Private Schools*. Q&A 22-01. Revised February 2022. https://sites.ed.gov/idea/idea-files/q-and-a-children-with-disabilities-private-schools-parentally-placed/. Accessed 30 August 2026. No individual FAPE right; child find, consultation, proportionate share, services plans. Whether homeschoolers count depends on state definition of private school. [Cited]

University of Pittsburgh Office of Admissions. "Home Schooled Applicants." https://admissions.pitt.edu/home-school/. Accessed 30 August 2026. Minimum 3 years mathematics among other Carnegie units; Home School Supplemental Form. Named page, not NACAC policy. [Cited]

Redford, Jeremy, Danielle Battle, and Stacey Bielick. *Homeschooling in the United States: 2012*. NCES 2016-096. Washington, DC: NCES, 2016. ERIC ED569947, https://files.eric.ed.gov/fulltext/ED569947.pdf. Accessed 30 August 2026. 1.773 million adjusted 2012 estimate; 88% of HS-level homeschoolers taught basic algebra. Do not narrate 88 → 79 as decline without checking standard errors. [Cited]

Reich, Rob. "The Civic Perils of Homeschooling." *Educational Leadership* 59, no. 7 (2002). https://ascd.org/el/articles/the-civic-perils-of-homeschooling. Accessed 30 August 2026. Civic/democratic critique, not a math-outcome study. [Cited]

Southern Methodist University Admission. "SMU Application Guide for Homeschool, Non-Traditional Students." https://www.smu.edu/admission/apply/undergraduate-admission/home-schooled-nontraditional-students. Accessed 30 August 2026. Minimum *suggested* three units mathematics: Algebra I, Geometry, Algebra II; textbook titles in course descriptions. Pattern, not a rule. [Cited]

Southwest Baptist University Admissions. "Homeschool College Admission." https://sbuniv.edu/admissions/homeschool-students.php. Accessed 30 August 2026. 3 units mathematics recommended; parent or organization transcript; ACT/SAT/CLT. Test-optional for admission; scores may affect scholarships. [Cited]

Swaner, Lynn E., Albert Cheng, and Jonathan Eckert. *School-Sector Influence on*

Graduate Outcomes and Flourishing: Findings from the 2023 Cardus Education Survey. Cardus, 2024. https://www.cardus.ca/research/education/reports/school-sector-influence-on-graduate-outcomes-and-flourishing/. Accessed 30 August 2026. Companion CES 2023 report. Think tank, representative panel, noncausal. [Cited]

4. AI products, evidence, and law

Oreopoulos / Low / Hamilton County Schools. AEA RCT Registry AEARCTR-0013519. 2024. https://www.socialscienceregistry.org/trials/13519. Accessed 30 August 2026. Pre-registration for the Khanmigo trial. [Cited]

Anthropic. "Introducing Claude for Teachers." 14 July 2026, updated 28 August 2026. https://www.anthropic.com/news/claude-for-teachers. Accessed 30 August 2026. Educators only, 18+, FERPA DPA, no training on teacher chats. Not a student companion. [Cited; Vendor]

Bastani, Hamsa, et al. AsPredicted 4DL_Q3J (pre-registration). 2023. https://aspredicted.org/4DL_Q3J. Accessed 30 August 2026. Primary analysis = unaided exam. Cited in Bastani PDF. [Cited]

Bastani, Hamsa, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman. "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics." *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. https://doi.org/10.1073/pnas.2422633122. Author PDF, https://hamsabastani.github.io/education_llm.pdf. Accessed 30 August 2026. Field RCT, ~1,000 Turkish high-school mathematics students, GPT-4-0613. GPT Base +48% practice / −17% unaided exam versus control; GPT Tutor +127% practice / −0.004 NS on exam. GPT Base fully correct 51% on the practice set (42% logical / 8% arithmetic errors). Percentages of control grades, not Cohen's *d*. Not a homeschool study. [Cited]

California Legislature. A.B. 1159 bill status. https://leginfo.legislature.ca.gov/faces/billStatusClient.xhtml Accessed 30 August 2026. 30 August 2026: Senate third reading, in floor process. Do not treat as law. [Cited]

California Legislature (Addis). A.B. 1159, Student personal information. 2025–2026 session. Bill text as amended 21 August 2026, https://leginfo.legislature.ca.gov/faces/billNavClient.x

Accessed 30 August 2026. NOT enacted as of 30 August 2026. Would ban using covered student information to train generative AI. [Cited]

Chervonyi, Yuri, et al. / Google DeepMind. "Gold-medalist Performance in Solving Olympiad Geometry with AlphaGeometry2." arXiv:2502.03544, 2025. https://arxiv.org/html/2502.03544v1. Accessed 30 August 2026. Actual arXiv title (mid-2 errata). Specialized neuro-symbolic system; 84% (42/50) of IMO 2000–2024 geometry versus 54% for AG1. Not a kitchen chatbot result. Remaining authors not locked from a title-page refetch this pass. [Cited]

Cursor. "Cloud Agents." https://cursor.com/docs/cloud-agent. Accessed 30 August 2026. Developer cloud VMs; formerly Background Agents. Distinct from Grok Bot teammate UX. [Cited; Vendor]

Cursor. "Plans and billing (Grok Bot)." https://cursor.com/help/grok-bot/plans. Accessed 30 August 2026. Individual versus Teams access. [Consulted; Vendor]

Cursor. "Privacy and data." https://cursor.com/help/security-and-privacy/privacy. Accessed 30 August 2026. DPA for Teams/Enterprise; individual plans follow Privacy Policy. Not FERPA. [Consulted; Vendor]

Epoch AI. "FrontierMath Tiers 1–3 (v2)." https://epoch.ai/benchmarks/frontiermath-tiers-1-3-v2. Accessed 30 August 2026. 338 problems after 12 June 2026 v2; OpenAI funding conflict disclosed. Live accuracy % not extracted from the plot this day. [Cited; UNVERIFIED live accuracy]

Escueta, Maya, Andre Joshua Nickow, Philip Oreopoulos, and Vincent Quan. "Upgrading Education with Technology: Insights from Experimental Research." *Journal of Economic Literature* (2020). https://doi.org/10.1257/jel.20191507. Cited inside Oreopoulos and Low for a 0.05–0.20 SD CAL range. Not re-opened in full this day. [Consulted; UNVERIFIED full text this pass]

Federal Trade Commission. "Children's Online Privacy Protection Rule" (final amendments). 90 FR 16918 (22 April 2025). Document 2025-05904, 16 CFR 312. https://www.federalregister.gov/documents/2025/04/22/2025-05904/childrens-online-privacy-protection-rule. GPO HTML, https://www.govinfo.gov/content/pkg/FR-2025-04-22/html/2025-05904.htm. Accessed 30 August 2026. Effective 23 June 2025; general compliance 22 April 2026. Ed-tech / school-authorization amendments not finalized. [Cited]

Garcia, Megan. *Garcia v. Character Technologies et al.* Complaint. M.D. Fla. CourtListener/RECAP, https://storage.courtlistener.com/recap/gov.uscourts.flmd.433581/gov.uscourts.fln

Accessed 30 August 2026. Allegations re Sewell Setzer III (age 14) and Character.AI. Not a verdict. Settlement terms UNVERIFIED this day. [Cited]

Google. “Google Workspace for Education — homeschool setup.” https://support.google.com/edu/setup/a Accessed 30 August 2026. Homeschool org domain path. Gemini Education SKU details UNVERIFIED on a dedicated product page this day. [Cited; Vendor]

Center for AI Safety / Scale AI / HLE Contributors. “Humanity’s Last Exam.” https://lastexam.ai/. Accessed 30 August 2026. Snapshot 30 August 2026: Gemini 3 Pro 38.3%, GPT-5 25.3%, Grok 4 24.5%, Claude 4.5 Sonnet 13.7%, GPT-4o 2.7% overall. Math subset not broken out. Do not update from later aggregator scores. [Cited]

InternGeometry authors. “Achieving Olympiad-Level Geometry LLM Agent via Complexity Boosting RL.” arXiv:2512.10534, 2025. https://arxiv.org/abs/2512.10534. Research preprint 44/50 IMO geometry. Not a home product. [Cited; Preprint]

Kestin, Greg, Kelly Miller, Anna Klales, Timothy Milbourne, and Gregorio Ponti. “AI Tutoring Outperforms In-Class Active Learning: An RCT Introducing a Novel Research-Based Design in an Authentic Educational Setting.” *Scientific Reports* (2025). https://www.nature.com/articles/s41598-025-97652-6. Harvard undergraduates, physics, custom tutor, immediate post-test. Post M = 4.5 vs 3.5; ~0.63 SD (linear; underestimate due to ceiling). Not K–12 math, not the kitchen default. Numbers from secondary HTML of the paper. [Cited]

Kosmyna, Nataliya, et al. / MIT Media Lab. “Your Brain on ChatGPT: Accumulation of Cognitive Debt when Using an AI Assistant.” arXiv:2506.08872, 2025. https://arxiv.org/abs/2506.08872. EEG/essay study, not math-specific. Optional supporting citation for offloading; not a math RCT. Not named on this manuscript’s notes. [Consulted; Preprint]

Liang, Weixin, Mert Yuksekgonul, Yining Mao, Eric Wu, and James Zou. “GPT Detectors Are Biased against Non-Native English Writers.” arXiv:2304.02819, https://arxiv.org/abs/2304.02819; *Patterns* 4, no. 7 (2023): 100779, https://doi.org/10.1016/j.patter.2023.100779. Accessed 30 August 2026. Average false-positive rate on 91 human TOEFL essays: **61.22%** (arXiv) / **61.3%** (*Patterns*). Book keeps 61.22 and reports both. Near-perfect on 88 U.S. eighth-grade essays. Do not use detectors to police a homeschooler’s written math. [Cited]

Microsoft. “Math Solver (Math Assistant / Math Progress).” https://www.microsoft.com/en-us/research/project/math-solver/. Accessed 30 August 2026. OneNote Math Assistant and Teams Math Progress — M365 Education, not a confirmed consumer Copilot Math SKU. Standalone app status UNVERIFIED. [Cited; Vendor]

Nous Research. "Configuring Models | Hermes Agent." https://hermes-agent.nousresearch.com/docs/user-guide/configuring-models. Accessed 30 August 2026. Model picker and auxiliary models. [Consulted; Vendor]

Nous Research. "Hermes 4 — Qwen-3 14B" (model-card example). 2025. https://huggingface.co/NousResearch/Hermes-4-14B-FP8. One size of the Hermes 4 family, not the Agent runtime. [Consulted; Vendor]

Nous Research. "Hermes 4 Technical Report." arXiv:2508.18255, 2025. https://arxiv.org/abs/2508.18255. HTML conversion failed 30 August 2026; MATH / GSM8K metrics UNVERIFIED pending PDF. Hermes 4 is not Hermes Agent. [Cited; Preprint; UNVERIFIED metrics]

Nous Research. "Hermes Agent — Open-Source AI Agent with Persistent Memory." https://hermes-agent.org/. Accessed 30 August 2026. MIT license; February 2026 release; no-telemetry claim is vendor. Parent-machine tool, not a child-safety school product. GitHub: https://github.com/nousresearch/hermes-agent/. [Cited; Vendor]

Nous Research. "Nous Portal | Hermes Agent." https://hermes-agent.nousresearch.com/docs/integrations/
portal. Accessed 30 August 2026. Hermes 4 is not recommended inside Hermes Agent. [Cited; Vendor]

OpenAI. "A Free Version of ChatGPT Built for Teachers." 19 November 2025. https://openai.com/index/chatgpt-for-teachers/. Accessed 30 August 2026. Teacher-facing, U.S. K–12 verified, free through June 2028, FERPA-oriented workspace. Not a student math SKU. [Cited; Vendor]

OpenAI. "Bringing ChatGPT for Teachers to More U.S. School Districts." 26 August 2026. https://openai.com/index/bringing-chatgpt-for-teachers-to-more-us-school-districts/. Accessed 30 August 2026. Still teacher-facing. [Consulted; Vendor]

Oreopoulos, Philip, and Nina Low. "One Click Away: AI Tutoring with Khanmigo in a Two-Year School Experiment." NBER Working Paper 35620, August 2026. https://www.nber.org/papers/w35620 and PDF https://www.nber.org/system/files/working_papers/w35620/w35620.pdf. Accessed 30 August 2026. Second author Nina Low (not Nathan). Working paper, not yet journal-peer-reviewed. Two-year cluster RCT, 18 Tennessee middle schools. Table 5 ITT: 1.264 NPR/term (SE 0.597); **0.040 population SD per term**. Abstract 1.3 NPR/term and 0.06–0.08 SD over a *school year* — not standalone ITT. Gains "resemble those from Khan Academy practice without AI assistance." Median student messaged Khanmigo on a third of practice days and in 17% of mistake sessions; 14.5% of messages contained a mathematical question or a step of reasoning. [Cited; Preprint / working paper]

Pane, John F., Beth Ann Griffin, Daniel F. McCaffrey, and Rita Karam. "Effectiveness of Cognitive Tutor Algebra I at Scale." *Educational Evaluation and Policy Analysis* 36, no. 2 (2014). https://doi.org/10.3102/0162373713507480. Pre-LLM ITS (Carnegie Cognitive Tutor). Cited in Oreopoulos and Low. Historical, not Hermes. [Cited]

Phan, Long, and colleagues / CAIS / Scale. "A Benchmark of Expert-Level Academic Questions to Assess AI Capabilities." *Nature* 649 (2026): 1139–46. https://www.nature.com/articles/s41586-025-09962-4. Accessed 30 August 2026. Nature paper for Humanity's Last Exam. Remaining authors not locked from a title-page refetch this pass. [Cited]

Raine, Matthew, and Maria Raine. *Raine v. OpenAI*. Complaint. 2025. https://s3.documentcloud.org/documents/26078538/raine-vs-openai.pdf. Accessed 30 August 2026. Allegations re Adam Raine and ChatGPT. Not a verdict. [Cited]

Stanford HAI. *Artificial Intelligence Index Report 2025*. https://aiindex.stanford.edu/. Accessed 30 August 2026. 2025 snapshot (GSM8K saturation, FrontierMath ~2% then). Stale relative to 30 August 2026 model scores; historical context only. [Cited]

xAI. "xAI and El Salvador Pioneer the World's First Nationwide AI Education Program." 11 December 2025. https://x.ai/news/el-salvador-partnership. Accessed 30 August 2026. Grok-as-tutor (the assistant, not Grok Bot) in public schools. Announcement, not RCT. Independent evaluation UNVERIFIED. [Cited; Vendor]

xAI / Cursor. "Grok Bot FAQ." https://docs.x.ai/grok-bot/faq. Accessed 30 August 2026. Cloud storage required; Legacy Privacy Mode unsupported; shared computer; no education claims. "Grok Bot requires cloud data storage, so Legacy Privacy Mode is not supported." [Cited; Vendor] xAI. “Grok Bot Is Now Included with More Plans.” 26 August 2026. https://x.ai/news/grok-bot-more-plans. Accessed 30 August 2026. SuperGrok + Cursor Pro/Teams. [Cited; Vendor]

xAI / Cursor. “Grok Bot for Teams and Enterprises.” https://docs.x.ai/grok-bot/teams-and-enterprises. Accessed 30 August 2026. One Linux VM per member; Privacy Mode (Legacy) blocks Grok Bot. [Consulted; Vendor]

xAI. “Introducing Grok Bot.” 11 August 2026. https://x.ai/news/introducing-grok-bot. Accessed 30 August 2026. Early beta; cloud computer; not an education SKU. [Cited; Vendor]

---

5. Vendor, catalogue, advocacy, journalism (labelled)

Art of Problem Solving. “Beast Academy Books.” https://beastacademy.com/books. Accessed 30 August 2026. Levels 1–5, four units A–D each. Vendor. [Cited; Vendor]

Art of Problem Solving. “Beast Academy FAQ (books).” https://beastacademy.com/faq/books. Accessed 30 August 2026. “We generally recommend that students start about a level ‘behind’ their current grade level.” Loosely based on CCSS; Level 5 prepares for AoPS Prealgebra. Vendor. [Cited; Vendor]

Art of Problem Solving. Beast Academy pricing. https://www.beastacademy.com/pricing. Accessed 30 August 2026. Pricing table did not render a public dollar table on 30 August 2026 fetch. Exact list price UNVERIFIED this access day. [Cited; Vendor; UNVERIFIED list price]

Balingit, Moriah. “How a True Believer’s Flawed Research Helped Legitimize Home Schooling.” *Washington Post*, 11 December 2023. https://www.washingtonpost.com/education/202 ray-homeschool-student-outcomes/. Accessed 30 August 2026. Journalism. Quotes Kunzman on Cardus and on the non-significance of homeschooling for achievement. Not a dataset. [Cited; Journalism]

Boaler, Jo. “Timed Tests and the Development of Math Anxiety.” Stanford GSE / *Education Week*, 2012. https://ed.stanford.edu/news/boaler-timed-tests-and-development-math-anxiety. Accessed 30 August 2026. Opinion / commentary, not a journal experiment. Labelled advocacy plus citation of anxiety/working-memory literature. [Cited; Advocacy]

Boaler, Jo. *Fluency Without Fear: Research Evidence on the Best Ways to Learn Math Facts*. Youcubed working paper, 2015. https://www.youcubed.org/evidence/fluency-without-fear/. Accessed 30 August 2026. Working paper / Youcubed publication, not a journal experiment. [Cited; Advocacy; working paper]

Carnegie Learning. "MATHia by Carnegie Learning." https://www.carnegielearning.com/solutions/math/ Accessed 30 August 2026. School ITS grades 6–12. RAND/EMERALDS summaries are vendor-cited. Direct family purchase UNVERIFIED 30 August 2026. [Cited; Vendor; UNVERIFIED family purchase]

Carnegie Learning. "Meet MATHia." https://discover.carnegielearning.com/meet-mathia. Accessed 30 August 2026. Vendor marketing. Same school-facing product. [Consulted; Vendor]

Cathy Duffy Reviews. "CTC Math." https://cathyduffyreviews.com/homeschool-reviews-core-curricula/math/math-grades-k-6/ctc-math. Accessed 30 August 2026. Catalogue/reviewer, not an RCT. Publisher-provided access; affiliate disclosure. Homeschool pricing quoted: family $198.50/12 months; individual $128.50/12 months after homeschool discount. Confirm on vendor site before printing. [Cited; Vendor / catalogue]

CoinDesk. "El Salvador Partners with Elon Musk's Grok in AI-Powered Education for 1M Students." 11 December 2025. https://www.coindesk.com/tech/2025/12/11/el-salvador-partners-with-elon-musk-s-grok-in-ai-powered-education-for-1m-students. Accessed 30 August 2026. Journalism on the xAI announcement. Not independent evaluation. [Cited; Journalism]

Community College Daily (American Association of Community Colleges). "DataPoints: New Data on Dual Enrollment." August 2024. https://www.ccdaily.com/2024/08/datapoints-new-data-on-dual-enrollment/. Accessed 30 August 2026. Journalism reporting first-year IPEDS dual-enrollment counts (2.5 million; 73.5% at public community colleges). Not a peer-reviewed paper. [Cited; Journalism]

Coalition for Responsible Home Education. "Homeschooling & Disabilities." https://crhe.org/guides/resources-for-homeschool-parents/disabilities/. Accessed 30 August 2026. Advocacy (children's-rights). Accurate on the FAPE gap; prescriptive IEP-at-home advice is CRHE's stance, not a federal requirement.

[Cited; Advocacy]

Coalition for Responsible Home Education. “Research with an Agenda (Ray 2010).” https://crhe.org/homeschooling-academics-and-demographics-ray-2010/. Accessed 30 August 2026. Advocacy critique of Ray/HSLDA *Progress Report*. Useful dissection of the 84th-percentile math claim. Not a re-analysis of raw data. [Cited; Advocacy]

DreamBox / Discovery Education. “DreamBox Math for Homeschool.” https://www.dreambox.com/family/homeschool-math. Accessed 30 August 2026. K–8 family SKU. ESSA Strong and “one hour / grade level” are vendor claims; underlying study PDF not opened. [Cited; Vendor; UNVERIFIED ESSA PDF]

DreamBox / Discovery Education. DreamBox (main). https://www.dreambox.com/. Accessed 30 August 2026. K–8 adaptive math/reading. Not a generative agent. [Cited; Vendor]

Education Week. “Scholars Craft New Approaches to Teaching Fractions.” 2013. https://www.edweek.org/teaching-learning/scholars-craft-new-approaches-to-teaching-fractions/2013/08. Accessed 30 August 2026. Journalism summarizing NMAP teacher survey and Fuchs/Siegler/Jordan work. Not a paper. [Consulted; Journalism]

Houghton Mifflin Harcourt / Saxon Homeschool. *Middle Grades Math Placement Test*. https://images.salsify.com/image/upload/s–hxkiZwTr–/60b1c78e60f3fc141b8ed703c0c62f98d9b68153.pdf. Accessed 30 August 2026. Score bands into Math 5/4 through Algebra 1. Publisher: the test is not infallible; Saxon is skill-level not grade-level. [Cited; Vendor]

Houghton Mifflin Harcourt. “Saxon Math Program.” https://www.hmhco.com/programs/saxon-math. Accessed 30 August 2026. Incremental approach. School-line ESSA evidence claims are about school implementations, not homeschool RCTs. [Cited; Vendor]

Houghton Mifflin Harcourt / Saxon. *Saxon Homeschool Math Scope and Sequence* (introduction PDF). https://www.hmhco.com/~/media/sites/home/education/global/pdf/scope-and-sequence/mathematics/k-12/saxon-math-homeschool/shs_ssintroduction.pdf?la=en. Accessed 30 August 2026. Recommended sequence Math K–3, 5/4, 6/5, 7/6, 8/7, Algebra 1, Algebra 2, Advanced Mathematics, Calculus; 80% mastery to advance. Year not on the PDF title this pass. [Cited; Vendor]

Homeschool Start Guide. “Dimensions Math vs Primary Mathematics: Which Singapore Math Curriculum for Homeschool?" https://homeschoolstartguide.com/blog/dimensions-math-vs-primary-mathematics. Accessed 30 August 2026. Secondary homeschool guide. Used only to flag edition-mixing warnings already confirmed on the vendor FAQ. [Consulted]

Home School Legal Defense Association. "California Homeschool Laws (at a glance)." https://hslda.org/legal/california. Accessed 30 August 2026. Three options; notification yes; teacher qualifications no except option 3; no assessment. [Cited; Advocacy]

Home School Legal Defense Association. "About the Evaluator Option." https://hslda.org/post/about-the-evaluator-option. Accessed 30 August 2026. Portfolio/evaluator alternative to standardized tests in states that allow it. [Consulted; Advocacy]

Home School Legal Defense Association. "Idaho Homeschool Laws (at a glance)." https://hslda.org/legal/idaho. Accessed 30 August 2026. No notification, no teacher qualifications, subjects yes, assessment no. [Cited; Advocacy]

Home School Legal Defense Association. "Illinois Homeschool Laws (at a glance)." https://hslda.org/legal/illinois. Accessed 30 August 2026. No notification, no teacher qualifications, subjects yes, assessment no. [Cited; Advocacy]

Home School Legal Defense Association. "Massachusetts Homeschool Laws (at a glance)." https://hslda.org/legal/massachusetts. Accessed 30 August 2026. Notification yes; assessment yes; teacher qualifications no; subjects yes. 2026 bills in flux — do not freeze pending legislation as law. [Cited; Advocacy]

Home School Legal Defense Association. "How to Comply with New York's Homeschool Law." https://hslda.org/post/how-to-comply-with-new-yorks-homeschool-law. Accessed 30 August 2026. Advocacy reading of 8 NYCRR § 100.10. IHIP, hours, quarterly reports, 33rd percentile/growth rule; SAT/ACT do not qualify. Confirm against NYSED regulation. [Cited; Advocacy]

Home School Legal Defense Association. "How to Comply with North Carolina's Homeschool Law." https://hslda.org/post/how-to-comply-with-north-carolinas-homeschool-law. Accessed 30 August 2026. One-time DNPE notice; instructor high-school diploma or equivalent; annual nationally standardized test that must measure mathematics. [Cited; Advocacy]

Home School Legal Defense Association. "North Carolina Homeschool Laws (at a glance).” https://hslda.org/legal/north-carolina. Accessed 30 August 2026. Notification yes; teacher qualifications yes; assessment yes. [Cited; Advocacy]

Home School Legal Defense Association. “Pennsylvania Homeschool Affidavit and Declaration.” https://hslda.org/post/pennsylvania-homeschool-affidavit. Accessed 30 August 2026. Annual notarized affidavit or unsworn declaration to superintendent by 1 August under option 1 (24 P.S. § 13-1327.1). [Cited; Advocacy]

Home School Legal Defense Association. “Pennsylvania Homeschool Laws (at a glance).” https://hslda.org/legal/pennsylvania. Accessed 30 August 2026. Four options; notification yes. [Cited; Advocacy]

Home School Legal Defense Association. “Special Education Provisions for Pennsylvania.” https://hslda.org/post/special-needs-provisions-for-pennsylvania. Accessed 30 August 2026. Citing 24 P.S. § 13-1327(d) and § 13-1327.1(e)(1). [Cited; Advocacy]

Home School Legal Defense Association. “How to Comply with Texas’s Homeschool Law.” https://hslda.org/post/how-to-comply-with-texas-homeschool-law. Accessed 30 August 2026. Required subjects include math; written curriculum per *Leeper*/private-school interpretation. [Cited; Advocacy]

Home School Legal Defense Association. “Texas Homeschool Laws (at a glance).” https://hslda.org/legal/texas. Accessed 30 August 2026. Advocacy, attorney-reviewed for members, not the Texas Education Code and not the *Leeper* opinion. [Cited; Advocacy]

Home School Legal Defense Association. “How to Comply with West Virginia’s Homeschool Law.” https://hslda.org/post/how-to-comply-with-west-virginias-homeschool-law. Accessed 30 August 2026. Annual assessment option: nationally normed test covering reading, language, mathematics, science, social studies; 4th stanine (23rd percentile) or improvement. [Cited; Advocacy]

International Center for Home Education Research Reviews (discussing Chelsea McCracken). “HOW TO MISLEAD WITH DATA: A Critique of Brian Ray’s Methodology.” https://icher.org/blog/?p=1440. Accessed 30 August 2026. Scholarly-adjacent blog (Kunzman’s ICHER). Not peer-reviewed; methodologically serious. [Cited]

InfoQ. “Google DeepMind’s AlphaGeometry2 AI Achieves Gold-Medal Math Olympiad Performance.” February 2025. https://www.infoq.com/news/2025/02/deepmind-alphageom2/. Accessed 30 August 2026. Journalism citing DeepMind: o1 and

Gemini Thinking solved none of the IMO-geometry set. Secondary. [Cited; Journalism]

IXL Learning. “IXL Math.” https://www.ixl.com/math. Accessed 30 August 2026. Skill lists Pre-K–Calculus. Adaptive practice, not an agent. Live homeschool price not on this URL. [Cited; Vendor; UNVERIFIED live invoice]

Khan Academy. “Meet Khanmigo.” https://www.khanmigo.ai/. Accessed 30 August 2026. Common Sense 4-star is vendor-cited. [Consulted; Vendor]

Khan Academy. “Khanmigo for Learners.” https://www.khanmigo.ai/learners. Accessed 30 August 2026. Parent signup; up to 10 children; U.S. billing. [Cited; Vendor]

Khan Academy. “Khanmigo Pricing.” https://khanmigo.ai/pricing. Accessed 30 August 2026. $4/month or $44/year for parents and learners; teachers free. Khan Academy content itself remains free. “Never gives you the answer” is vendor copy. [Cited; Vendor]

Khan Academy. “Khanmigo: Free, AI-Powered Teacher Assistant.” https://www.khanmigo.ai/teachers. Accessed 30 August 2026. Classroom student access only via district. Distinguishes teacher versus family product. [Consulted; Vendor]

Knowledge at Wharton. “Without Guardrails, Generative AI Can Harm Education.” 2025. https://knowledge.wharton.upenn.edu/article/without-guardrails-generative-ai-can-harm-education/. Accessed 30 August 2026. University journalism summarizing Bastani. Not a substitute for the paper. [Cited; Journalism]

Loeb & Loeb LLP. “Children’s Online Privacy in 2025: The Amended COPPA Rule.” 2025. https://www.loeb.com/en/insights/publications/2025/05/childrens-online-privacy-in-2025-the-amended-coppa-rule. Accessed 30 August 2026. Law-firm explainer. Secondary; defer to FR text. [Consulted]

Math-U-See. Placement tool. https://curriculumexpress.mathusee.com/. Accessed 30 August 2026. Yes/no mastery items; recommends a level. Not a standardized achievement test. [Cited; Vendor]

Demme Learning / Math-U-See. “Math-U-See Curriculum” sequence. https://mathusee.com/products/math-u-see-curriculum/. Accessed 30 August 2026. Official 13-level sequence Primer through Calculus. Epsilon is a fractions *level*, not “fifth grade.” Mastery / Build-Write-Say. [Cited; Vendor] mathed.net. “CCSSM domain/grade matrix.” https://mathed.net/ccssm.html. Unofficial HTML matrix. Fetch timed out 30 August 2026; used only as a search hit, not as a quoted primary. Year not on the page this pass. [Consulted; UNVERIFIED]

Mathway. “Mathway | Math Problem Solver.” https://www.mathway.com/. Accessed 30 August 2026. Type or photo; freemium. Vendor solver. [Cited; Vendor]

McGraw Hill. “ALEKS K–12.” https://www.mheducation.com/prek-12/program/aleks/MKTSP-GAB02M0.html. Accessed 30 August 2026. Knowledge Space Theory adaptive practice. Mastery-rate claims are vendor. [Cited; Vendor]

McGraw Hill. ALEKS portal. https://www.aleks.com/. Accessed 30 August 2026. Independent Use / family library mentioned in marketing copy. [Cited; Vendor]

Ray, Brian D., Braden Hoelzle, and Douglas Pietersma. “How Many Homeschool Students Are There in the United States During the 2024-2025 School Year?” National Home Education Research Institute, 27 February 2026. https://nheri.org/how-many-homeschool-students-are-there-in-the-united-states/. Accessed 30 August 2026. Advocacy research institute. Estimate 3.408 million (6.262%) via Pulse plus 23-state registers plus assumed undercount. Not NCES. [Cited; Advocacy]

National Home Education Research Institute / *Home School Researcher*. “The Impact of Homeschooling on Math Education.” https://nheri.org/home-school-researcher-the-impact-of-homeschooling-on-math-education/. Accessed 30 August 2026. Advocacy journal, small convenience sample. Correlation of <1 hour/day math with lower computation scores. NOT a national time-on-task study. [Cited; Advocacy]

Pew Research Center. “A Look at Homeschooling in the U.S.” 20 February 2025. https://www.pewresearch.org/short-reads/2025/02/20/a-look-at-homeschooling-in-the-us/. Accessed 30 August 2026. Journalism restating NCES PFI 2023. Adds 1.8% full-time virtual not considered homeschool. Not a primary survey. [Cited; Journalism]

Photomath. Photomath homepage. https://photomath.com/. Accessed 30 August 2026. Scan-to-solution solver; $0 / $9.99 mo / $69.99 yr displayed 30 August 2026. “Designed for learning” is vendor. No Photomath-named unaided-transfer RCT opened. [Cited; Vendor]

Rainbow Resource Center. "CTCMath Family Membership (12 month subscription)." https://www.rainbowresource.com/007424.html. Accessed 30 August 2026. Family membership listed at $397 pre-discount. Not an effectiveness study. [Cited; Vendor / catalogue]

Rainbow Resource Center. "Saxon Math Homeschool Curriculum K-12." https://www.rainbowresource.com/saxon-math. Accessed 30 August 2026. Vendor catalog. Notes that classic high-school Saxon integrates geometry inside Algebra 1, Algebra 2, and Advanced Mathematics. Confirm against HMH. [Cited; Vendor / catalogue]

Ray, Brian D. / National Home Education Research Institute. *Homeschool Progress Report*. 2009 (HSLDA-hosted) and 2010 academic write-up. Advocacy. Specimen: about the 86th percentile overall, **84th in mathematics**, 89th in reading; convenience sample recruited via testing services and networks; disproportionately White, Christian, two-parent, higher-income, higher parental education. Publisher/place/page details not independently catalogued this pass. [Cited; Advocacy; UNVERIFIED catalog details]

Singapore Math Inc. "School FAQ." https://www.singaporemath.com/pages/school-faq. Accessed 30 August 2026. Dimensions Math 6 "includes some Pre-algebra topics, but is not a Pre-algebra text in the traditional sense; it is a Grade 6 text." Dimensions 7+8 cover pre-algebra and Algebra 1 with some geometry. [Cited; Vendor]

Singapore Math Inc. "Placement Tests." https://www.singaporemath.com/pages/placement-tests. Accessed 30 August 2026. Untimed placement PDFs by series/edition; publisher instructs not to use the test as the sole factor. [Cited; Vendor]

Symbolab. "Symbolab AI Math Calculator." https://www.symbolab.com/. Accessed 30 August 2026. Vendor solver plus practice/quizzes/worksheets. [Cited; Vendor]

Teaching Textbooks. Homepage. https://www.teachingtextbooks.com/. Accessed 30 August 2026. Claims "100% of the teaching" and "100% of the grading." Price table not displayed on homepage 30 August 2026 — list price UNVERIFIED on this primary. Official scope-and-sequence PDF not opened. [Cited; Vendor; UNVERIFIED list price]

Texas CBE. "AP Calculus AB: The Complete Guide for 2026 & the May 2027 Exam." 2026. https://www.texascbe.com/blog/ap-calculus-ab-2026-complete-guide. Secondary commercial guide. Facts taken from College Board primaries, not from this page. [Consulted]

Transparency Coalition. “In New Lawsuit, Parents Allege ChatGPT Responsible for Their Teenage Son’s Suicide.” 2025. https://www.transparencycoalition.ai/news/parents-of-suicidal-teen-sue-openai-over-chatgpts-role-in-sons-death. Accessed 30 August 2026. Advocacy journalism summarizing the Raine complaint. Use the complaint as primary. [Cited; Advocacy / Journalism]

University of Chicago News. “Parents’ Math Anxiety Can Undermine Children’s Math Achievement.” 2015. https://news.uchicago.edu/story/parents-math-anxiety-can-undermine-childrens-math-achievement. Accessed 30 August 2026. University news-office summary of Maloney et al. Journalism-adjacent; the finding is the journal article. [Cited; Journalism]

Wolfram Research. “Wolfram|Alpha: Computational Intelligence.” https://www.wolframalpha.com/. Accessed 30 August 2026. Algorithms/knowledgebase, not an LLM. CAS check, not an explainer. Pro step-by-step price not on homepage this day. [Cited; Vendor; UNVERIFIED Pro price]