Math for Little Thinkers

Michael Gannotti

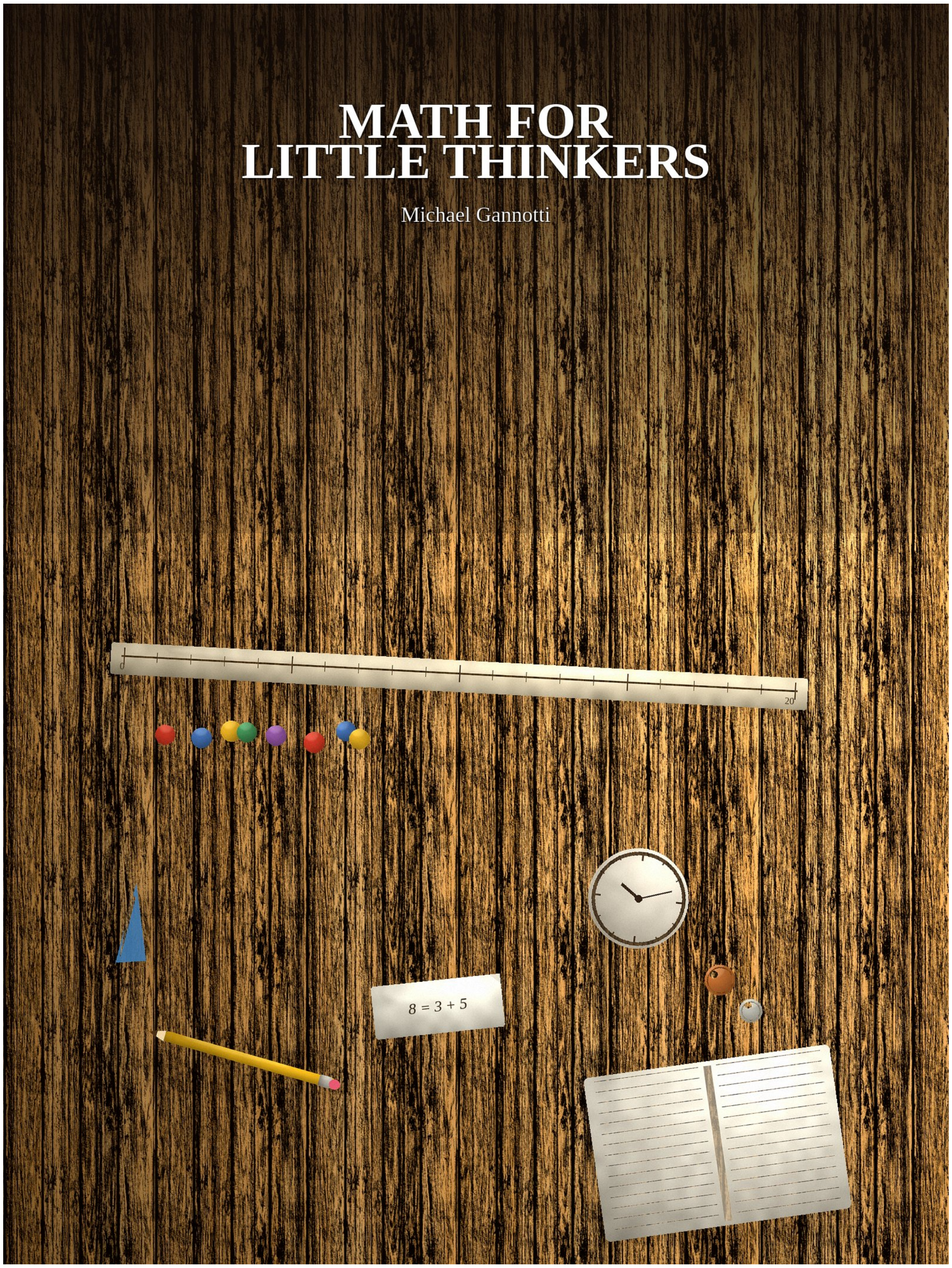

Figure 1: Cover

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This illustrated edition is the teaching-manual of 3 September 2026. Endnotes run in one series. Chapter-opening images are original still-lifes, not portraits of persons or children. Documents still constrain the facts.

Contents

Welcome

This book exists because the math hour at your table is the whole program. There is no math department down the hall and no specialist waiting after lunch. There is you, a five- to ten-year-old, and today's idea — counters or a line, a question worth waiting for, a number sentence they write. That is enough, if you know what to do with the hour.

I wrote this for a capable, busy, willing parent of a five- to ten-year-old. You may be teaching two ages at once, five-to-seven and eight-to-ten, at one table. You may be fitting a math sit between a job, a toddler, and a grocery run. You may love number, or you may remember elementary math as a fog of worksheets. Many adults feel rusty when they sit down to teach. That feeling is common. It is not a verdict. We will move on from it. This book will make you fluent enough in today's idea to hear a wrong turn, and to ask a good question — without taking the pencil.

You do not need to be a mathematician. You do need to hear `1/2 + 1/3 = 2/5` as a size mistake, not a cute slip. You need to hear `8 + 4 = □ + 5` answered as 12, a pizza drawing offered as the number 3/4, "altogether" glued to addition, a recitation to 100 offered as number sense, and a timed worksheet offered as the first fluency move. You need a session shape you can run on a Tuesday. You need a few sentences that actually help. That is the job. A path through ordinary challenges — counting, the equal sign, equal groups, fractions on a line, measure, money, shapes, story-problem types — is possible at a kitchen table. The path is not a personality trait and it is not a percentile. It is a small set of moves, practiced on number the child already somewhat knows, with you in the chair.

About 3.4 percent of U.S. students ages 5–17 were homeschooled in 2022–23 —

roughly 1.765 million children.[1] That figure is context, not a ranking. You are one of those tables. Many readers of this book will not be in that pond. They will be after-school, weekend, or kitchen-table parents of enrolled children. The moves still fit. Homeschooling is legal in all fifty states and the District of Columbia. The paperwork is not one load: Texas asks almost nothing of the state agency; New York asks for a written plan and regular reports.[2] There is no national homeschool diploma and no national "Little Thinkers Math I" credit.[3] None of Texas, North Carolina, Pennsylvania, Virginia, or New York requires a critical-thinking course. Math, or arithmetic, is the usual required or assumed elementary object.[4] In the last federal tables that listed subjects taught at home, arithmetic was the math row:

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[1]Ellena Sempeles and Jiashan Cui, *Parent and Family Involvement in Education: 2023*, NCES 2024-113 (Washington, DC: National Center for Education Statistics, September 2024), Table A-6: 3.4 percent homeschooled, approximately 1,765,000 students, ages 5–17 with a K–12 grade equivalent, 2022–23; kindergarten–2nd grade 4.0 percent; 3rd–5th grade 3.6 percent. https://nces.ed.gov/pubs2024/2024113.pdf. Access date for URLs in these notes: 3 September 2026. Latest federal count this book uses; not a 2026 national total. A neighboring 5.2 percent received instruction at home (homeschooled or full-time virtual) and is a different bucket.

[2]Texas Education Agency, "Home Schooling," https://tea.texas.gov/families-and-students/finding-school-your-child/home-schooling. New York State Education Department, "Home Instruction Questions and Answers," https://www.nysed.gov/nonpublic-schools/home-instruction-questions-and-answers. Homeschooling is legal in all fifty states and D.C.; these two pages illustrate a light-process and a heavier-process pattern, not a national template. This book is not legal advice.

[3]Texas Education Agency, "Home Schooling": "The State of Texas does not award a diploma to students that are home schooled." New York State Education Department, "Home Instruction Questions and Answers": a high school diploma "may only be awarded to a student enrolled in a registered secondary school." North Carolina Division of Non-Public Education, "Home School Records Retention & Diplomas FAQs," https://www.doa.nc.gov/divisions/non-public-education/home-schools/faqs/records-retention-diplomas: the State does not issue a diploma or a transcript; the home-school chief administrator does. "Little Thinkers Math I" is not a credit in any opened statute.

[4]Texas Education Agency, "Alternative Schooling," restating *Leeper*: bona fide instruction using a written curriculum of reading, spelling, grammar, math, and a course in good citizenship. North Carolina Division of Non-Public Education, "Home School Requirements and Recommendations": annual nationally standardized test in English grammar, reading, spelling, and mathematics. Pennsylvania 24 P.S. § 13-1327.1(c)(1): elementary shall include English, arithmetic, science, geography, history, civics, safety education, health and physiology, physical education, music, and art. Virginia Department of Education, "Home Instruction": the parent writes a list of subjects; evidence of progress is a composite score, SAT/ACT/PSAT, or an evaluation letter. New York 8 NYCRR 100.10(e)(2)(i): grades 1–6 required subjects are arithmetic, reading, spelling, writing, English language, geography, United States history, science, health, music, visual arts, and physical education. None of these five opened maps names a critical-thinking course. Math or arithmetic is the usual required or assumed elementary object. The parent reads their own statute.

83 percent of kindergarten-through-second-grade homeschoolers and 86 percent of third-through-fifth-grade homeschoolers that year.[5] The hour in front of you is the work.

This book is not a reprint. *Mathematics for Homeschooling* covers grades 1–12. It compressed elementary into two chapters and then ran through algebra, geometry, statistics, and calculus. One pointer, if you need the later years later. Then we teach at five to ten. This book is not a sequel. *Critical Thinking Skills for Little Thinkers* is the same house, a different subject — reasons, testimony, changing a mind. One pointer, then we teach this age's math.

What a good math hour looks like

You sit down already knowing today's idea. An object is on the table — counters, a number line, coins, a clock, a measuring cup. The child warms up on facts they can already get right. You model one problem out loud, short. Then they try, and you wait. You ask one good question — not "did you get it?" but "how many are there, really?" or "what does the equal sign mean here?" Three seconds feels long. It is the work. Practice mixes the new move with last week's. The hour ends with one or two items they do alone. You stop talking sooner than feels polite. The child talks and builds. You hold the key. That is the hour. The next long piece of this front matter, *The Math Hour*, will teach it in full. Later chapters will not reinvent it.

[5] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey*, NCES 2020-001 (Washington, DC: National Center for Education Statistics, 2019), Table 9: arithmetic taught that year, 83 percent of K–2 homeschoolers and 86 percent of 3–5 homeschoolers (64 percent of all K–12). The published subject lists include arithmetic, basic algebra, geometry, and neighboring subjects. Quote the arithmetic row. The 2023 First Look did not republish the subject-taught tables. https://nces.ed.gov/pubs2020/2020001.pdf.

Figure 2: A kitchen-table math hour

Children this age already do mathematics. They do not wait for a readiness gate, and they do not become number-sense-ful because they can recite to 100.[6] A child who answers 12 for `8 + 4 = □ + 5` is applying a prior idea of the equal sign, not displaying an absence of math. Pizza can introduce a share. The number, when we are done, lives on a line.

[6]National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Finding 8: "Most children acquire considerable knowledge of numbers and other aspects of mathematics before they enter kindergarten." Finding 15: claims that children of particular ages cannot learn certain content because they are "too young," "not in the appropriate stage," or "not ready" have consistently been shown to be wrong if prerequisite knowledge is present. Finding 10: conceptual understanding, computational fluency, and problem-solving skills belong together. https://files.eric.ed.gov/fulltext/ED500695.pdf.

What you will actually get

Each teaching chapter does eight jobs, always in the same order, so you are never hunting for the move.

You will learn why this week's idea is worth the struggle. You will understand it yourself, in plain language, with the wrong answers you should be able to hear. You will get a session you can run this week: exact wording, named try-its in the home and outside it, and a talk box. Your student will get a short section of their own. If it isn't clicking, you will get three diagnostics and a next move, with no shame in the room. Tools, including AI, stay optional and adult-side. And you will get a plain checklist for "done enough," so you can place by skill rather than by birthday.

The eight teaching chapters follow the work, not twelve thin grade labels: counting and number sense; place value; addition, subtraction, and the equal sign; fluency; equal groups; fractions as numbers on a line; measurement, time, and money; shapes, early data, and story-problem types. Records and resources come last.

This week you can learn the session shape and today's idea well enough to hear a size mistake. Today the child can count a pile, write a number sentence, or place a mark on a line.

What this book will not do

This book will not hand you 180 days of worksheets. A year of photocopies is not a teaching method, and I will not pretend it is. Coverage is a map. Depth is the hour.

It will not sell you a curriculum. Later, a short resources chapter names common programs by fit — parent load, style, how they place a student — so you can choose. Combining does not turn a brand into a course name. The transcript still says Mathematics or Arithmetic.

It will not treat a timed-test kit as fluency, a pizza forever as fractions, or a keyword list as story problems. It will not franchise bar models as a miracle. A picture of the story is a representation, not a proof that one publisher won.

It will not promise a score. There is no guaranteed percentile, no math diploma, and no certificate of fluency inside these pages. What this book can promise is a path: the ideas in order, at the skill the child actually has, until they can do the next one unaided.

It is not "Little Thinkers Math I." It is not a sequel to *Autonomous AI and Education*. One math-hour AI box lives in the next long piece. Later chapters point back.

It is not a culture-war pamphlet. Safe math objects only: counters, a number line, coins, a clock, shapes, a recipe half-cup, a store price tag as a number, a walk that counts. A price tag is a number. A recipe is a half-cup. Live fights stay off this table.

And this is not a book that lectures your child about you, or you about your character. The student is a person, not a percentile. When this book speaks to them, it speaks with respect. No baby-talk. No research sermon.

The promise

If you only remember one sentence, remember this: a path through elementary mathematics is the promise. A diploma is not. A percentile is not.

The work is to take this child through mathematics they can actually use — how many, what a digit means, an equal sign that means the same as, equal groups, a fraction that lives on a line — without skipping the gate because they are "good at math," and without parking them in a workbook that is too easy because a catalog printed a grade on the cover. You will sometimes slow down. You will sometimes skip ahead. Both are teaching. Birthday is not placement.

Three things have to hold. You understand today's idea well enough to hear a wrong turn. The child attempts first, with objects, then a sketch, then a written equation. Any helper — including an AI tool — stays on your side of the table: a supplement you host and constrain, not a partner during the attempt, not a photo-to-key, and not a secret friend.

You can do this. You do not have to know next year's idea today. You have to know this week's idea well enough to sit still while they struggle, then ask one good question. Start here. Read *How to Use This Book This Week*, then the one-page list of five things, then *The Math Hour*. After that, open the chapter that matches the skill in front of you. You will know more after one chapter than you know this morning. Your child will have something to try today. We can do this.

How to Use This Book This Week

Start at the skill in front of you, not on page one because a catalog, a birthday, or a well-meant relative said so.

This book is a handbook you open to this week's idea, not a novel you read cover to cover. The eight teaching chapters follow the work, not twelve thin grade labels. Records and resources come last. Age bands here are 5–7 and 8–10. They tell you the grain of the same ideas. They do not tell you where *this* child sits.

This book does not replace the program already on the shelf. If a math curriculum, a kitchen number line, a library counting book, or a store hour you already use is working, keep it. Use this book to hear a wrong turn, to run the hour, and to name the work a stranger can read. Combine honestly. The title is still Mathematics or Arithmetic. Never a brand. Never Little Thinkers Math I.

Two ages at one table

You may be teaching a five-to-seven-year-old and an eight-to-ten-year-old in the same sit. That is ordinary. The session shape in *The Math Hour* still holds; the object on the table and the grain of the question change.

The younger child talks and builds. Counters, a spoken how-many, a number sentence such as `8 = 3 + 5` — that is enough. The check is near: the pile, the line, the coins. The older child can hold a little more: equal groups, a fraction on a 0–1 line, a story-problem type named in ordinary words. You may run a twenty-minute count with one child and a thirty-minute fraction sit with the other. You do not need two personalities. You need two first problems.

Pick the chapter by skill, not by birthday

Open the chapter you think is right. Skip to **What "done enough" looks like** at the end of the *previous* chapter, or to **If it isn't clicking** in the one you opened. If the child can already do those checks unaided, you are too early. If the checks from two chapters back are still failing, drop back.

If they still live in reciting without cardinality — if they can chant to 100 and still fail to produce a set of 8 — they are still in the 5–7 band of the counting chapter, whatever their age. Children this age already do mathematics. What they can do depends on what they have already been invited to try, not on a birthday.[7]

If they cannot yet place 1/2 on a 0–1 line, they are still in a sharing-to-number stage, whatever the birthday. A nine-year-old who still treats a pizza drawing as the number is not "behind in fractions." They are still learning that the fraction is a location.

If they can already count with meaning, write `8 = 3 + 5` as true, and place a unit fraction on a line, and they are still in a "grade 2 math workbook" only because the cover says so, skip ahead. Instruction for a six-year-old need not be "first grade"; it should match what the student actually needs and already knows.[8] A publisher's

[7] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Finding 15: claims that children of particular ages cannot learn certain content because they are "too young," "not in the appropriate stage," or "not ready" have consistently been shown to be wrong if prerequisite knowledge is present. The Panel quotes National Research Council, *Taking Science to School*, ed. Richard A. Duschl, Heidi A. Schweingruber, and Andrew W. Shouse (Washington, DC: National Academies Press, 2007), p. 2: "What is developmentally appropriate is not a simple function of age or grade, but rather is largely contingent on prior opportunities to learn." Access date for URLs in these notes: 3 September 2026. https://files.eric.ed.gov/fulltext/ED500695.pdf. Douglas H. Clements and Julie Sarama, "Learning Trajectories in Early Mathematics — Sequences of Acquisition and Teaching," *Encyclopedia on Early Childhood Development*, revised August 2023: ages in trajectory tables are rough guides; Reciter is not yet Corresponder and is not yet Counter (cardinality). https://www.child-encyclopedia.com/pdf/expert/numeracy/according-experts/learning-trajectories-early-mathematics-sequences-acquisition-and.

[8] New York State Education Department, "Home Instruction Questions and Answers," https://www.nysed.gov/nonpublic-schools/home-instruction-questions-and-answers: instruction for a six-year-old need not be "first grade"; "instruction should be geared to the level appropriate to the student's needs and previous level of achievement." Common Core State Standards for Mathematics (Washington, DC: National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010): "These Standards do not dictate curriculum or teaching methods"; what students can learn at a grade "depends upon what they have learned before." A map of

grade label is a scope, not a legal grade. Use this book's checklists, then teach.

How you use the parent half

Most of each teaching chapter is for you. Read it *before* the lesson, not over the child's shoulder.

Why this matters tells you what this idea unlocks. **For the parent: understand it yourself** gives one everyday picture, one precise picture, and three to five wrong answers you should be able to hear. Sit with those.

How to teach it this week assumes the session shape from *The Math Hour*. It will not rebuild the hour. It will give you this week's named try-its — at least one in the home and one outside it — the wording for *this* idea, and a talk box: an exact opening question, three to five follow-ups, how to wait, and what a stuck silence usually means. Later chapters supply this week's questions. They do not reinvent the shape.

Practice that actually builds learning names the try-its with time, materials, safety, the fun, and the skill. The kitchen, the walk, the store, and the library *are* the practice. They still have to land on the number line and the written equation. **Tools, including AI** is optional, short, and for the adult. The rules live once in *The Math Hour*. **What "done enough" looks like** is how you leave: unaided work, not a perfect Tuesday.

You do not have to read the whole chapter tonight. You do have to read the parent half of *this week's* idea before you sit down with the child.

"Lesson plans" in this book means this week's session shape. "Assessment tests" means a short skill check, an exit ticket, a done-enough checklist, and optional publisher placement. Neither is a 180-day dump. Neither is a percentile battery.

The five-minute parent warm-up

Five minutes. Child not yet in the chair. Phone face down.

1. Read today's idea until you can say it in one sentence.

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typical U.S. public-school placement, not a homeschool statute. http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf.

2. Look at one object yourself — the counters, the line, the coins. Ask, out loud, what would count as a check. Stay off any tool until you have looked.
3. Glance at the “wrong answers you should be able to hear.” Name the one you would have given at fourteen.
4. Write one sentence you will actually say. Not a speech. Example: “How many are there, really?” Or: “What does the equal sign mean here?”
5. Close the book to the student page. You are ready.

If you are learning the idea *while* they are stuck, you will talk too much. Prepare first. Then sit still.

How the student uses “For the student”

Every teaching chapter includes a short section written to the student, not about them. One or two pages. Warmer. Direct. What the idea is, a tiny worked example, two tries, an “explain it back” prompt, and one challenge. Hand it over after your short model, not instead of it. You stay in the room.

The student page is not something to send off with an unsupervised chatbot. The attempt is still theirs. You still hold the key.

In the home and outside it

Every teaching chapter names try-its at the table and try-its on a walk, at a store, on a playground, or at the library. Kitchen, money, and making motivate. They do not replace the number line or the written equation.

When to skip ahead

Skip ahead when this chapter’s “done enough” checklist is already true *unaided*. Slow down when the same wrong turn repeats after a clear look. **If it isn’t clicking** will give you three likely causes and a next move. Hearing a wrong answer, then asking a better question, then naming a better move if needed, is teaching.

For this week: pick the chapter by skill. Do the five-minute warm-up. Run the hour as *The Math Hour* describes it. Let the student page be theirs. Stop talking sooner than you want to.

If You Only Remember Five Things

Keep this page. The chapters will add wording, objects, and this week's questions. They will not replace these.

1. They already do math. Children this age already do mathematics. Counting to 100 is not number sense. Reciting a verbal sequence is not how-many. A child who can chant to one hundred and still fail to produce a set of 8 is still learning cardinality, not waiting for a readiness gate. Start now, with objects they can move, then a sketch, then an equation. You do not wait for a stage. You do not wait for a diploma.[9]

2. Number line and written equation, not pizza forever. Pizza can introduce a share. The number, when you are done, lives on a line. Write the equation. `8 = 3 + 5` is true in the 5–7 band. `3/4` is a location, not a drawing that never becomes a number. Word problems are types of action and relation — join, separate, compare, equal groups — not keyword lists. "Altogether" does not mean add.[10]

[9]National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Finding 8: most children acquire considerable knowledge of numbers before kindergarten. Finding 15: "too young" / stage claims wrong if prerequisites are present. Access date for URLs in these notes: 3 September 2026. https://files.eric.ed.gov/fulltext/ED500695.pdf. Douglas H. Clements and Julie Sarama, "Learning Trajectories in Early Mathematics — Sequences of Acquisition and Teaching," *Encyclopedia on Early Childhood Development*, revised August 2023: Reciter (verbal sequence) is not yet Corresponder (one-to-one) and is not yet Counter (cardinality). https://www.child-encyclopedia.com/pdf/expert/numeracy/according-experts/learning-trajectories-early-mathematics-sequences-acquisition-and. Douglas Frye et al., *Teaching Math to Young Children*, NCEE 2014-4005 (Washington, DC: Institute of Education Sciences, November 2013), Recommendation 1, moderate evidence: subitize small collections, then one-to-one counting, then compare, then numerals, then simple problems. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/early_math_pg_111313.pdf.

[10]National Mathematics Advisory Panel, *Foundations for Success*, Finding 10: concep-

3. Child attempts first. The student talks and builds. You ask one good question and wait. Three seconds is a convention, not a sacrament.[11] After they stop, wait again. If the silence is hard, look at the counters, the line, or the coins, not at the child. Struggle before rescue: ask, wait, hint, then model. You do not grab the pencil. You do not finish the item. A helper, including an AI tool, stays on your side of the table. It does not sit in the chair during the attempt.

4. Fluency is accurate, then sometimes timed. Retrieve known facts often. Time them only after the child can already get them right untimed, for one to five minutes, as one piece of the hour — not the lesson, and not a page of the same item against a clock.[12] Automatic recall frees working memory for the next idea. A kitchen timer used to rank a seven-year-old is not fluency.

5. Kitchen, money, and making motivate. They do not replace the line or the

tual understanding, computational fluency, and problem-solving skills belong together. Finding 12: a key mechanism linking conceptual and procedural knowledge of fractions is the ability to represent fractions on a number line. Robert Siegler et al., *Developing Effective Fractions Instruction for Kindergarten through 8th Grade*, NCEE 2010-4039 (Washington, DC: Institute of Education Sciences, September 2010), Recommendation 2, moderate evidence: fractions are numbers; use number lines as a central representational tool. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/fractions_pg_093010.pdf. Eric J. Knuth, Ana C. Stephens, Nicole M. McNeil, and Martha W. Alibali, "Does Understanding the Equal Sign Matter? Evidence from Solving Equations," *Journal for Research in Mathematics Education* 37, no. 4 (2006): 297–312: many students view = as announcing a result; relational ("the same as") as best definition was 32 percent of sixth graders, 43 percent of seventh, 31 percent of eighth — older than this book's ceiling, which is why `8 = 3 + 5` belongs now. Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*, WWC 2021006 (Washington, DC: Institute of Education Sciences, 2021), Recommendation 5, strong evidence, 18 studies: teach word-problem types, not keyword-to-operation matching; Table 5.2, *altogether* fails on equal groups. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/WWC2021006-Math-PG.pdf.

[11] Robert J. Stahl, "Using 'Think-Time' and 'Wait-Time' Skillfully in the Classroom," ERIC Digest ED370885 (1994), https://files.eric.ed.gov/fulltext/ED370885.pdf: think-time; about three seconds as a convention; written for classroom teachers. Mary Budd Rowe, "Wait-Time and Rewards as Instructional Variables," NARST paper, ERIC ED061103 (1972), https://files.eric.ed.gov/fulltext/ED061103.pdf, is elementary science class, not a homeschool trial and not a math randomized trial. This book uses the pause so already-present reasoning can be heard.

[12] Fuchs et al., *Assisting Students Struggling with Mathematics*, Recommendation 6, strong evidence, 27 studies: timed activities last 1 to 5 minutes; add them once students have been working on a concept over many lessons; do not use them to introduce concepts; the panel does not recommend merely giving timed worksheets. Population: grades K–6 intervention for students struggling, not a parent-hour trial. National Mathematics Advisory Panel, Finding 11: automatic recall of facts frees working memory for more complex problem solving.

equation. A recipe half-cup, a store price tag, a walk that counts, a snack shared equally — those are reasons to sit down. The work still lands on the number line and the written equation. Everyday stakes. No live controversy on the table. If a stranger asks what you taught, the title is Mathematics or Arithmetic, not a brand. A path through elementary mathematics is the promise. A diploma is not. A percentile is not. Birthday is not placement.

The child attempts first. You hold the key. If this week needs a compass, this is it. Five things. Then sit down and teach.

The Math Hour

The hour has a shape. Learn it once. Later chapters will give you today's idea, today's try-it, and today's questions. They will not rebuild this hour. When a chapter says "run the session," it means this.

You do not need a school bell. You need a warm-up, a short model, a real attempt, one good question then wait, mixed practice, and an unaided exit. A five-to-seven-year-old may finish nearer twenty minutes; an eight-to-ten-year-old may need forty. The shape does not change.

Sit down having already done the five-minute parent warm-up from *How to Use This Book This Week*. You know today's idea well enough to hear `1/2 + 1/3 = 2/5` as a size mistake. The child talks and builds. You hold the key. Twenty to forty minutes, most weekdays, plus one named try-it outside the home, is the right ambition.[13] There is no national table of homeschool math minutes.

Life of the habit, as a session rhythm, lives here: revisit yesterday; mix last week; stop.

[13] Douglas Frye et al., *Teaching Math to Young Children*, NCEE 2014-4005 (Washington, DC: Institute of Education Sciences, November 2013), Recommendation 5, minimal evidence: dedicate time each day to teaching math and integrate math throughout the day; the panel still judged daily dedicated time worth recommending. Access date for URLs in these notes: 3 September 2026. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/early_math_pg_111313.pdf. Session length in this chapter is pedagogical design for the kitchen table, not a national math-minutes study (none was found). North Carolina DNPE *recommends* (not law) five hours across all subjects; Pennsylvania statute names 180 days or 900 elementary hours for the whole program. None of these is a national math block.

1. Warm-up

Two to five minutes. Known facts, oral or on a mini number line. Already-right material. This is retrieval, not a test of character.

Say: “We’re going to start with things you already know.”

Ages 5–7: making 10. “Tell me two ways to make 10.” Ages 8–10: a factor family. “What’s 6 $\times$ 8? Take your time.”

If they already get these right untimed, you may use a brief timer — one to five minutes, not the lesson.[14] Say: “We’ll time this only because you already get these right. Ready?” If they do not already get them right, skip the timer. Fluency is accurate first, then sometimes timed. You are building retrieval, not an identity.

Keep this short. The warm-up is not the lesson.

2. Short model

Three to seven minutes. One worked example, think-aloud, objects or line visible, then the equation. Then you stop. Show the move. Then fade.[15]

Say: “I’m going to show this one short. Then you’ll try.” “Watch. The equal sign means the same amount on both sides. That’s why I can write `8 = 3 + 5`.”

[14] Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*, WWC 2021006 (Washington, DC: Institute of Education Sciences, 2021), Recommendation 6, strong evidence, 27 studies: timed activities last 1 to 5 minutes; they are not the entire focus; add them once students have been working on a concept over many lessons; do not use them to introduce concepts; the panel does not recommend merely giving timed worksheets. Population: grades K–6 intervention for students struggling, not a parent-hour trial. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/WWC2021006-Math-PG.pdf. National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Finding 11: automatic recall frees working memory for more complex problem solving.

[15] National Mathematics Advisory Panel, *Foundations for Success*, Finding 27: explicit instruction (clear models, extensive practice, think-alouds, extensive feedback) has consistently positive effects for students with mathematical difficulties and the lowest third, on word problems and computation; “this does not mean that all of a student’s mathematics instruction should be delivered in an explicit fashion.” Finding 23: high-quality research does not support instruction that is entirely student-centered or entirely teacher-directed. Fuchs et al., Recommendation 1, strong evidence: systematic, incremental instruction with review of prior content.

On the second pass, leave a hole. When the idea is new, show one incorrect example — 12 for `8 + 4 = □ + 5`, or `1/2 + 1/3 = 2/5` — and ask what went wrong. If you are still talking at minute eight, close the model.

3. Student attempt

This is the center of the hour. Eight to fifteen minutes. One to four items of *today's type*. Concrete, then a sketch, then a written equation, as needed. Hand them "For the student" or the first try-it. Then you talk less than you want to.

Say: "This one is yours. I'll be quiet." Then be quiet. If they stall, use this order: ask, wait, hint, then model. Not the reverse. Ask: "How many are there, really?" Wait. Count a slow three. The silence is the work. If you fill it, you took the problem back.

Hint, one hint: "You already know we can touch each and say one number. Point to the pile." Then, if they are still stuck after a real try: "I'm going to show you this one step. Then you take it from here." Keep your hands off their pencil. The child talks and builds. You hold the key.

4. One good question, then wait

One to three minutes of clock time that feels longer. The talk around it can run using the talk-box shape below. Not "did you get it?"

An authentic question is one for which you have not already written the answer. "What is the vocabulary word?" is recitation. Say: "What does the equal sign mean here?" Or: "Where does this fraction live on the line?"

Then wait. A slow three after you ask. A slow three again after they stop. Robert Stahl called that silence *think-time*; that label is classroom-origin, not a homeschool trial. About three seconds is a convention, not a sacrament.[16] Mary Budd Rowe found the same pause in elementary science class. That was science class.

[16] Robert J. Stahl, "Using 'Think-Time' and 'Wait-Time' Skillfully in the Classroom," ERIC Digest ED370885 (1994), https://files.eric.ed.gov/fulltext/ED370885.pdf: eight categories of classroom silence; about three seconds as a convention. Classroom-origin. Not a finding that wait-time teaches mathematics, and not a homeschool randomized trial.

This book uses the pause so already-present reasoning can be heard.[17] If they are mid-count or mid-reason, do not cut them off. If the silence is hard, look at the counters, the line, or the coins, not at the child. One question. Maybe a follow-up. Then stop.

5. Mixed practice

Five to ten minutes. Yesterday and last week mixed with today. Not forty of the new item.[18]

Say: “Two from today. One from last week. Then we stop.”

Kitchen, money, and making can sit inside this mix. They do not replace the line or the equation.

6. Exit ticket

Two to four minutes. Two to four items, one of them yesterday’s skill. Book closed. No hints. No chatbot. Say: “These few are yours. I won’t help. That’s the point.” Done-enough is “right, or wrong-with-a-reason we can use tomorrow,” not a percentage.

After you ask, wait. If you fill the silence, you took the problem back. Praise a clear count, a mark on the line, or a mind changed after a recount — not speed. Hands in your lap. If a wrong answer keeps coming, hear it, ask a better question, then name a better move if needed. Hearing a wrong answer and leaving it uncorrected is not kindness.

[17] Mary Budd Rowe, “Wait-Time and Rewards as Instructional Variables: Their Influence on Language, Logic, and Fate Control,” NARST paper, ERIC ED061103 (1972), https://files.eric.ed.gov/fulltext/ED061103.pdf: elementary science classes; teachers typically waited about one second after a question and 0.9 seconds after a student answer; extending both to three to five seconds changed discourse. Rowe is science class. This book does not cite Rowe 1972 as a homeschool randomized trial or as a math trial.

[18] Fuchs et al., Recommendation 1: mix previously and newly learned material. Recommendation 5, strong evidence, 18 studies: teach word-problem types (Change / Combine / Compare / equal groups), not keyword-to-operation matching; Table 5.2, *altogether* fails on equal groups (4 cartons of 12 eggs is not 4 + 12).

If the hour goes sideways

Use this as a debug, not a verdict.

- Child guesses the operation from a word (“altogether,” “left,” “each”) → ask what kind of story this is; draw the situation; try the trap once with numbers they can see.
- Child cannot start → smaller numbers, objects, re-model one.
- Child is accurate untimed, slow → short known-fact retrieval; later, a brief timer.
- Child is fast and wrong → untimed accuracy; delay the clock.
- Child waits for rescue → count three; sit on your hands.
- Child stuck after a real try → one think-aloud, then a similar item.
- Session is only worksheets → put one item on the line; mix yesterday.
- Parent feels rusty and reaches for the pencil → put the object back; let them build and say.[19]

Revisit yesterday. Mix last week. Stop. That is the habit. It is not a diploma.

The talk-box shape, once

Later chapters will fill this week’s opening question. They will not reprint this shape. When a chapter prints a talk box, it means this.

Opening question. Authentic. About how many, what a digit means, what the equal sign means, which facts are already known, whether this is equal groups, where a fraction lives on the line, what is being measured, or what kind of story this is. No answer you have already written. Not “did you have fun” and not “what is the vocabulary word.” Locked openings live in the teaching chapters — how many, what a digit means, what the equal sign means, which facts are known, equal groups, where a fraction lives, what is being measured, what kind of story this is.

[19] Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock, “Intergenerational Effects of Parents’ Math Anxiety on Children’s Math Achievement and Anxiety,” *Psychological Science* 26, no. 9 (2015): 1480–88. First- and second-graders of math-anxious parents learned less math over the year only when those parents reported frequent homework help; the association was math-specific and disappeared when such parents helped less often. School homework-help study, not a homeschool trial. The portable caution: let the child build and say.

Three to five follow-ups. Pick; you do not need all five every Tuesday. Meaning: What does that number mean? Representation: Show me with things, a sketch, the number line, or a number sentence. Type or magnitude: Is this equal groups, or something else? Is this more than one half? Check: Does that answer make sense? Mind-change: You may change your mind. Try it another way.

How to wait. After you ask, a slow three. After they stop, wait again. If they are mid-count or mid-reason, do not cut them off. Look at the counters, the line, or the coins, not at the child, if the silence is hard.

What a stuck silence usually means. The question was vague. Wait-time after the question was zero. They are hunting a keyword. They are guessing what you want. The numbers are too big. They offered a recitation where cardinality was asked for. They think “=” means “the answer comes next.” They think “altogether” means add. Pose, wait, and point back at the object. You do not stack five questions or switch to a live controversy.

The in-home try-it shape, once

Later chapters will fill named types. They will not reprint this spine. When a chapter says “run the in-home try-it,” it means this.

Time. A short sit, then stop. Five minutes at the counter. Fifteen with a measuring cup. Pedagogical design, not a national minutes table.

Materials. Household objects: pasta, a measuring cup already in the drawer, a clock already on the wall, coins, a board-game score pad. You do not need a kit.

Safety. Ordinary: coins and small counters are a choking hazard for the youngest. No flame for a five-year-old. Ordinary food hygiene. Tone: curious, not courtroom. Let the child move the pasta.

The fun. Counting eggs. Sharing a snack. Making a dollar. Reading the clock until something starts.

The skill. Cardinality. Join or separate. A unit fraction. Place value. Elapsed time as a line analogue. Equal groups with a leftover named.

Types you will meet, not a lesson bank: kitchen count; recipe half or double; clock / “how long until”; money jar; board-game scores; sharing a snack equally, leftover named. A kitchen fair share uses a cup already in the drawer. It is not a nutrition sermon.

The out-of-home try-it shape, once

"Outside" means the walk, the store, the playground, the library, the stairs. Later chapters will fill named types. They will not reprint this spine.

Types you will meet: store money (5–7, coins to pay; 8–10, change — never "altogether means add"); walk shapes, including skinny triangles and a square on a vertex; playground measure (5–7, steps or sticks; 8–10, a tape); stairs or elevator; bus or parking numbers; sharing a bakery pack. A walk is number, shape, and measure you were already going to meet.

Safety, once. Roads, water, aisle courtesy. No shoplifting a "test." No lecture to a stranger's child. No public quiz of a cashier. A price tag is a number. It is not a sermon about a food system.

The library is a free math hour. In the last federal tables that asked, two-thirds of homeschool households used the library as a source of curriculum and books.[20]

This week, said plainly

One focus skill. Four or five math hours of the session shape, numbers adjusted to the child. Mixed review of last week's skill inside every hour. One in-home try-it and one out-of-home try-it that use the same type. Friday: look at the week's exit tickets; pick one diagnostic wrong answer; decide whether next week repeats, narrows, or moves on. That is the honest answer to "lesson plans," not 180 days written in advance.

[20] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey*, NCES 2020-001 (Washington, DC: National Center for Education Statistics, 2019), Table 6: 66 percent of homeschool households used the library as a source of curriculum and books in 2016 (K–2, 78 percent; 3–5, 70 percent). The 2023 First Look did not republish that table. Ellena Sempeles and Jiashan Cui, *Parent and Family Involvement in Education: 2023*, NCES 2024-113, Table A-5: 53 percent of the Homeschool-total row visited a library in the past month, versus 29 percent of all K–12 — a visit, not a curriculum-source percent.

A short skill check is three objects, not a battery: an exit ticket of two to four items, one of them not today's new skill; one diagnostic wrong answer the child explains; a done-enough checklist by skill, not by grade. Optional fourth: a publisher placement test after a gap — not a yearly identity.

The AI rules, once

Later chapters will point here in one sentence. They will not reprint this box. The child talks to the parent. The parent may use a tool. The child does not sit alone with an open chat. The math is on the table: counters, a number line, coins, shapes, a recipe half-cup. The child attempts first. The parent holds the answer key.

Ages 5–7: no live open chat as the child's partner. You may generate a four-line SCRIPT for after the attempt, and you may ask a tool to explain a named page to you. No child-facing chat. No child account.

Ages 8–10: still parent in the room. The child may sit next to you while you use a tool. They may hear a hint after they have tried. They may not have their own companion account. They may not be asked to "prompt the tutor until it agrees." Unaided first. SCRIPT after the try.

You may, on your account, after the child has tried:

- Explain this idea *to you* from a named lesson (title, page, or today's object: counters, number line, coins, shapes, a measuring cup). You still have to understand it well enough to hear a wrong turn.
- Extra isomorphic practice with the answer key held by you. Same structure, new numbers. The child never sees the key. Do not generate "practice" by photographing tonight's assigned worksheet.
- A labelled SCRIPT after the try — a short spoken sequence for *you* to say, not a paragraph for the child to copy as their work. Example: "Touch each. Say one number. The last number you said is how many."
- A hint after an attempt. A hint is a question or a named idea, not the finished number. "Are the jumps the same size?" is a hint. "The answer is 12" is not.

- Diagnose work the child already produced. You hear the diagnosis and decide. Crop to the paper. Do not upload the child's face.

You may not:

- Paste the worksheet and ask the tool to complete it.
- Ask "what's the answer" or "just give me the steps" during the child's attempt, or let the child ask that of a window.
- Photo-to-key. Point a camera at the page (Photomath, Mathway, Symbolab, or any scan-and-solve) so a solution pops up for the exact problem. That loop is a solver. Photo-to-answer is the ban.
- Park an unsupervised chatbot as the child's partner during the attempt — not at 5, not at 7, not at 10. No "math friend" account.
- Write the student's work. The model does not fill "I counted ___ because ___."
- Invent facts (how many counters are in the pile, whether the recipe needs a half-cup, what a coin is worth, what the number line on the page actually shows). Look.
- Grade the child's work from a detector score.
- Print a certificate of fluency.
- Hand the child an agent (Hermes Agent, Grok Bot, or any cloud-computer worker) as a math partner.

In a high-school math trial, an unguarded chatbot made practice look better and left students worse on the test with the window closed.[21] This is a useful study, not a promise that every home will see the same result. That is not a 5–10 trial. The kitchen-table rule it supports: the model may prepare the adult and the next problem; it may not do the child's problem.

Federal practice guides and the National Mathematics Advisory Panel

[21] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without guardrails can harm learning: Evidence from high school mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. Field experiment, nearly 1,000 Turkish high-school math students, grades 9–11: a ChatGPT-like tool raised assisted practice 48 percent relative to control, then cut unaided exam grades 17 percent; a guarded tutor that withheld the answer raised practice 127 percent and left the unaided exam about the same as control. This is a high-school math trial, not a 5–10 trial. The transferable caution is the crutch.

name counters, number lines, word-problem types, fluency sprints, whole numbers, fractions, and certain geometry and measurement. They do not name a chatbot as the math teacher.[22]

Detectors mislabelled more than half of some human essays as machine-written.[23] Do not police a seven-year-old's "I counted 8" with a score. Language models invent sources.[24] If you asked a tool for a paper, open the paper. IES and NMAP are public PDFs.

Names, in ordinary language. Hermes 4 is a model. Hermes Agent is an agent. Grok is a chat assistant. Grok Bot is a cloud-computer agent. Game artist makes pictures. None of these is a 5–10 math class. Neither is a counting partner for a five-year-old.

COPPA covers children under 13. Ages 5–10 are under 13. The 22 April 2026 compliance date has passed.[25] Keep accounts, logs, and keys on the parent side.

Not a secret friend. A tool that talks in the first person is still a tool. It does not get a bedroom, a private channel, or a promise to keep secrets from the parent.

You can teach this entire book with no AI. The hour still has the same shape.

[22] Search of Frye et al., NCEE 2014-4005; Fuchs et al., WWC 2021006; Siegler et al., NCEE 2010-4039; and National Mathematics Advisory Panel, *Foundations for Success* (2008), for a recommendation to sit a child with a generative chatbot: it is not there. They name counters, number lines, word-problem types, fluency sprints, whole numbers, fractions, and certain geometry and measurement.

[23] Weixin Liang, Mert Yuksekgonul, Yining Mao, Eric Wu, and James Zou, "GPT detectors are biased against non-native English writers," *Patterns* 4, no. 7 (2023): 100779: average false-positive rate on human TOEFL essays 61.22 percent.

[24] William H. Walters and Esther Isabelle Wilder, "Fabrication and errors in the bibliographic citations generated by ChatGPT," *Scientific Reports* 13 (2023): 14045: 55 percent of GPT-3.5 citations and 18 percent of GPT-4 citations were fabricated in that study.

[25] Children's Online Privacy Protection Act, 15 U.S.C. §§ 6501 *et seq.*; 16 C.F.R. Part 312. Federal Register 90 FR 16918 (22 April 2025): amended Rule effective 23 June 2025; general compliance date 22 April 2026. Bright line: children under 13. Keep accounts on the parent side. Photomath homepage, fetched 3 September 2026: "A solution will pop up for your exact problem" sits next to "designed for learning" — the loop is the ban, not a trial. Khanmigo "never gives you the answer" is a design goal on a product page, not a 5–10 trial.

A Tuesday, said plainly

Here is an illustration, not a reported family. A parent has spent five minutes looking at a number line and counters. The child warms up on making 10. The parent shows `8 = 3 + 5` on the line, then puts the counters down. The child tries. The parent asks, "How many are there, really?" and waits. The child recites to eight. The parent hears the recitation, asks them to touch each, and they recount. Exit: "These two are yours. I won't help." On an eight-to-ten day the same shape holds, with a 0–1 line and a fraction to place.

You can run that hour. You need today's idea, a sentence you refuse to finish, and the willingness to stop talking. Later chapters will say: run the session as in *The Math Hour*.

Chapter 1

Counting and number sense

Figure 3: A kitchen-table still-life from a high three-quarter view: a 0–20 number line of tape on the table, a handful of pasta pieces in a row, a few counters in a small pile, a dated notebook with a blank line. No people. No logos. No readable package branding.

Why this matters

How many are there, really?

That is not a recitation question. It is not a test of whether the child can chant to one hundred. It is a question about a pile that is actually on the table — pasta, crackers, stairs, parked cars — and about whether the last number said names the pile.

A child can say the number words in order and still not know how many. They can touch each piece once and still start over when you ask again. They can chant to one hundred and still fail to make a set of eight, still fail to see that eight is more than five without recounting from one. Reciting a verbal sequence is one kind of work. Finding out how many is another.[26]

This chapter gives you one distinction you can keep for the rest of the book. Reciter is not Corresponder, and Corresponder is not Counter. Reciter says the words. Corresponder matches one word to one thing. Counter answers "how many?" with the last number counted, and means it.[27] Those names come from Douglas Clements and Julie Sarama. The ages they print beside the names are rough guides, not a calendar. High-quality experience can put a child a year or more ahead of a table average; without it, a child can sit years behind. Treat the names as what the child *does*, not as a birthday.[28]

What this idea unlocks is everything later. Place value needs a ten that is actually ten ones. Addition needs a start amount that is actually there. The equal sign needs two sides that name the same amount. Fractions will live on a number line — and the line is already a picture for whole numbers, this week, not a poster you save

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[26]National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, March 2008), Chapter 5, "Number Sense." In its most fundamental form, number sense is immediate identification of small quantities, facility with basic counting, and approximating magnitudes of small numbers of objects and simple numerical operations. Reciting a long verbal sequence is not that definition. ERIC ED500486. Access date for URLs in these notes: 3 September 2026.

[27]Douglas H. Clements and Julie Sarama, "Math, Science, and Technology in the Early Grades," *Encyclopedia on Early Childhood Development*, revised August 2023. Counting trajectory: Reciter (verbal sequence) is not yet Corresponder (one-to-one) and is not yet Counter (cardinality — answering "how many" with the last number counted). https://www.child-encyclopedia.com/

[28]Clements and Sarama (2023). Ages in trajectory tables "are rough guides only—children differ widely." "Children who are provided high-quality math experiences are capable of developing to levels one or more years beyond their peers." The ages are orientation, not goals.

for later.[29] If "number sense" still means a counting song, later pages arrive as symbol soup.

Two refusals belong in the same sitting as the distinction, or the distinction becomes a poster.

Children this age already do mathematics. The National Mathematics Advisory Panel put it plainly: most children acquire considerable knowledge of numbers, and of other parts of mathematics, before they enter kindergarten. Even in the preschool years, children have more reasoning and problem-solving ability than was long suspected.[30] Claims that a five-year-old cannot learn something because they are "too young," "not in the stage," or "not ready" have consistently been shown to be wrong when the prerequisite knowledge is present. What is appropriate is largely contingent on prior opportunities to learn.[31] Start now, with objects the child can move. A readiness gate is the wrong wait. Acceleration past a missing count is the other wrong wait.

Counting to one hundred is not number sense. In its most fundamental form, number sense is the ability to identify small quantities immediately, a facility with basic counting, and a feel for the size of small sets and simple operations. An advanced number sense, learned through instruction, needs place value, composing and decomposing whole numbers, and the meaning of the operations.[32] A chant to one

[29] National Mathematics Advisory Panel, *Foundations for Success* (2008), Chapter 5. Most five-year-olds can judge which of two single digits is larger, estimate the number of dots on a page, and determine the approximate location of single-digit numerals on a number line that provides only the numerical endpoints. The number line is already a whole-number representation in this band.

[30] National Mathematics Advisory Panel, *Foundations for Success* (2008), Learning Processes Finding 8: "Most children acquire considerable knowledge of numbers and other aspects of mathematics before they enter kindergarten." The same chapter: even during preschool, children have considerably greater reasoning and problem-solving ability than was suspected until recently.

[31] National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 15 / Chapter 5. Claims based on Piaget's theory, and related theories of "developmental appropriateness," that children of particular ages cannot learn certain content because they are "too young," "not in the appropriate stage," or "not ready" have consistently been shown to be wrong. NMAP quotes National Research Council, *Taking Science to School*, ed. Richard A. Duschl, Heidi A. Schweingruber, and Andrew W. Shouse (Washington, DC: National Academies Press, 2007), p. 2: "What is developmentally appropriate is not a simple function of age or grade, but rather is largely contingent on prior opportunities to learn." Piaget's primary texts were not independently opened for this book; the misapplication account is NMAP's.

[32] National Mathematics Advisory Panel, *Foundations for Success* (2008), Chapter 5, "Number Sense." Advanced number sense, acquired through formal instruction, "requires a principled understanding of place value, of how whole numbers can be composed and decomposed, and of the

hundred can sit on the Reciter level. It is not the check.

The Institute of Education Sciences early-math guide, for children ages three to six, gives the teaching order as Recommendation 1, with moderate evidence — the only recommendation in that 2013 guide that rose above minimal. First, recognize one, two, and three at a glance and name them without counting. Then count with one-to-one correspondence as a way of finding how many. Then compare. Then label with number words *and* numerals. Once those hold, solve simple problems.[33] That last step is the refusal of "only play." Play can be a setting. It is not a ceiling.

This book is not a reprint of *Mathematics for Homeschooling*. That book covers grades 1–12 and compressed elementary work into two chapters. One pointer, then we teach 5–10 math at the grain it actually needs. It is not a sequel to *Critical Thinking Skills for Little Thinkers*. Same house, different subject. One pointer, then we teach how many.

You do not need to be a mathematician. You do need to hear a recitation to one hundred offered as how-many, and to ask the cardinality question, without taking the pile.

This week you can learn to hear Reciter, Corresponder, and Counter as three different jobs. Today the child can count a small pile, say how many, and then produce a pile of that size.

For the parent: understand it yourself

Many adults feel rusty on the counting they last did as a child. That is ordinary. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The child still does the counting.

Everyday picture. Eight pieces of pasta on a plate. You can see them as a group

meaning of the basic arithmetic operations."

[33]Douglas Frye, Arthur J. Baroody, Margaret Burchinal, Sharon M. Carver, Nancy C. Jordan, and Judy McDowell, *Teaching Math to Young Children* (NCEE 2014-4005), Institute of Education Sciences, What Works Clearinghouse, November 2013. Recommendation 1, moderate evidence, ages 3–6: (1) subitize small collections (one to three) and label them with a number word without needing to count; (2) one-to-one counting as a means of identifying how many; (3) compare quantities; (4) label with number words and numerals; (5) once those fundamentals hold, solve basic problems. This is a useful study, not a promise that every home will see the same result. https://ies.ed.gov/ncee/wwc/practiceguide/18

without tagging each one, if the group is small enough. You can match one word to one piece. You can say "eight" and mean the plate, not the last piece. You can make another plate of eight. You can look at five and eight and say which has more without lining them up, or you can line them up. The plate is still eight. Nothing "comes next." The amount is the amount.

A second kitchen picture, still ordinary. Someone says, "I can count to one hundred." They do. The pasta is still on the plate. Ask how many. If they start at one again, or if they cannot make a set of eight, the chant was a song. Songs are useful. They are not how-many.

Precise picture. Three ideas sit under "count," and they are not the same.

Subitize. See a small collection at a glance and name it without counting. One, two, and three first. Then small sets to five. The IES guide wants this as step one: label the total with a number word *without needing to count them*.[34] A dice face of three, three crackers in a triangle, two shoes by the door — those are glances, not recitations.

One-to-one. Each object gets exactly one number word. Each number word gets exactly one object. Touch, move, or point. Keep track of what has been tagged. A line of objects is easier than a scatter. A scatter is easier than a moving flock of pigeons. The correspondence is the skill, not the speed.

Cardinality. The last number said is how many are in the set. After a count, if you ask "how many?" and they start over, the last word was a tag, not a name for the pile. After a count, if you say "give me eight" and they cannot produce the set, they can tag and still not make. Clements and Sarama split these: Counter (Small Numbers) counts a line to five *and* answers "how many" with the last number; Producer counts *out* a collection to five.[35] Both belong in the week.

Write those three on a scrap: glance; one word one thing; last number is how many. Circle the third. That is the question this chapter keeps asking.

A number line is already a picture for this work. Most five-year-olds, the Panel reports, can judge which of two single digits is larger, estimate a small number

[34] Frye et al., *Teaching Math to Young Children* (2013), Recommendation 1, how-to step 1.

[35] Clements and Sarama (2023), sample counting levels. Counter (Small Numbers): accurately counts objects in a line to 5 and answers "how many" with the last number counted. Producer (Small Numbers): counts out a collection to 5. Counter and Producer (10+): counts and counts out to 10, then toward 30.

of dots, and find the approximate location of a single-digit numeral on a line that shows only the endpoints.[36] Tape 0 and 10, or 0 and 20, on the table or the floor. "Stand on 7." "Who is closer to 10, 8 or 3?" The line will later hold fractions. It holds whole numbers now. Kitchen counters motivate. They do not replace the line, and they do not replace the written numeral.

Numerals belong beside the words. The IES order is: see small sets, count with meaning, compare, then label with number words *and* numerals, then simple problems.[37] "Eight" said aloud and `8` written are two names for the same pile. A child who can chant and cannot point to `8` still has a labeling job. A child who can write `8` and cannot make eight still has a cardinality job.

Compare is its own move. Once they can recognize or count collections, use number words and counting to decide which has more, which has fewer, whether they are the same. Matching one-to-one is a compare without numbers. Counting both and comparing the number words is a compare with numbers. Appearance — the longer row looks like more — is a trap you will hear. Line them up, or match, or count. Then write the two numerals.

Simple problems come last in that order, not never. "We had eight. We used two. How many now?" is already join and separate, with objects, then a sketch, then a number sentence when they can see it. Kindergarten on the common U.S. map already encourages equations such as `5 + 2 = 7`, and does not require the child to write them.[38] This chapter owns how-many. Chapter 3 owns the equal sign as "the same as." You may write `8` this week. You may write `8 = 5 + 3` when both sides are visible as the same pile. You need not wait for a prealgebra unit.

Clements and Sarama's sample counting levels, ages marked approximate, are a hearing aid, not a syllabus.[39] A child who chants in singsong, words not necessar-

[36]National Mathematics Advisory Panel, *Foundations for Success* (2008), Chapter 5, core number sense in five-year-olds, as in note 29.

[37]Frye et al., *Teaching Math to Young Children* (2013), Recommendation 1, steps 4–5.

[38]National Governors Association Center for Best Practices and Council of Chief State School Officers, *Common Core State Standards for Mathematics* (2010), Kindergarten overview. Students model simple joining and separating, "or eventually with equations such as 5 + 2 = 7 and 7 — 2 = 5." Student writing of equations in kindergarten is encouraged but not required. More learning time in Kindergarten should be devoted to number than to other topics. The Standards do not dictate curriculum or teaching methods and are used here as a map of typical U.S. public-school placement, not as a homeschool statute. http://www.corestandards.org/Math/

[39]Clements and Sarama (2023), Table 1 sample counting levels, ages marked approximate, with the authors' warning not to treat ages as goals.

ily separate or ordered, is still gathering the list. A Reciter says separate words, not always in the right order. Reciter (10) gets to ten, with some correspondence. A Corresponder matches words to objects, at least for small groups in a line. Counter (Small Numbers) counts a line to five and answers "how many" with the last number. Producer makes a set to five. Counter and Producer (10+) counts and counts out to ten, then toward thirty, keeps track in different arrangements, and treats the last number as how many. The table is orientation. Children differ widely. Plan to the level in the room, not to the birthday.

Wrong answers you should be able to hear

1. *"I can count to one hundred."* Offered as the answer to "how many are there?" A recitation offered as cardinality. Hear the song. Ask how many are *in this pile*. Then wait. The chant may be real and still not name the set.

2. *They finish "1-2-3-4-5," you ask "how many?," and they start over at one.* The last tag has not yet become the name of the set. The recitation was a performance. Hide the set and ask again. If they must restart, cardinality is still the job.

3. *They skip a piece, or tag one twice, or keep saying numbers after the pieces run out.* One-to-one is thin. The words may be in order. The matching is not. Slow the count. Touch. Move each piece as it is tagged. A line is kinder than a scatter.

4. *"Give me eight," and they hand you a handful, or they keep going past eight, or they stop at a piece that is not the eighth.* Producer is not yet Counter, or Counter is not yet Producer. They may answer "how many" after a count you led, and still not make the set. Count out together. Stop at eight on purpose. Then they do it.

5. *The longer row is "more," even when you can see the short row is bunched and the long row is spread.* Appearance is doing the work of compare. Match one-to-one, or count both, or push them into similar groups. Then ask again. The look of the row is not the number.

A sixth you will also hear: they subitize three when it is three, then insist on counting three every time after that, as if the glance were cheating. The glance is the first IES step. Counting is the second. Both are legal. "Did you count, or did you just see it?" is a good question, not a correction.

Five-minute parent warm-up

Do this before the lesson, with a small pile and a scrap of paper, no child in the room.

Minute 1. Put out eight objects. Glance at three of them. Say “three” without counting. Then count all eight, touching each. Say “eight.” That last word names the pile.

Minute 2. Ask yourself “how many?” without touching again. If you feel the urge to restart, you are rehearsing the miss. The last number still names the set.

Minute 3. Count out a set of five from a larger pile. Stop. That is Producer. Write 5. Write “five.” Both names.

Minute 4. Imagine the child saying “I can count to one hundred” when you asked how many pasta pieces. Decide, without performing, what a scold would do. Decide what “How many are there, really?” would do instead. You are installing the miss so you can hear it tomorrow.

Minute 5. Write the sentence you will actually say: “How many are there, really?” Put the pencil down. That sentence is the lesson.

If you can do those five minutes, you are ready to sit down. The child counts. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version, including the talk-box shape, the in-home and out-of-home try-it shape, and the AI rules. Here is the shape scaled to this idea.

Warm-up (2–5 minutes, unaided). A glance pile of one, two, or three. “How many? You can just see it.” Then one small count the child already does. No device. No chant-to-one-hundred as the warm-up.

Short model (3–7 minutes). One pile. You count, touching. You say the last number is how many. You write the numeral. You show the same amount on a 0–20 line. You stop. You are showing how-many, not delivering a lecture titled *Number Sense*.

Student attempt (8–15 minutes). One to four piles of *today’s size*. The child tags. The child says how many. The child produces a set. You wait. Struggle before rescue: ask, wait, hint (“touch each”), then a short model on *your* pile, not as their count.

One good question, then wait (1–3 minutes of clock time that feels longer). “How many are there, really?” Not “can you count to one hundred?” After you ask, a slow three is a convention, not a sacrament. After they stop, wait again. If they are mid-count, do not cut them off. Look at the counters, not at the child, if the silence is hard. Mary Budd Rowe measured those pauses in elementary *science* class, not at a kitchen table. Robert Stahl later called the silence think-time and wrote for classroom teachers generally. Use the move. You are not replicating a science-class study.[40]

Mixed practice (5–10 minutes). Yesterday’s small glance next to today’s count. A compare of two piles. A numeral to match. Mixing is how the child learns *when* to glance and *when* to count. Forty of the new item is not the mix.

Exit ticket (2–4 minutes). Two to four items. One pile: how many? One: give me this many. One from yesterday if you have it — a glance of three, or a compare. Done-enough is “right, or wrong-with-a-reason we can use tomorrow,” not a percentage.

That shape is a practice you impose on whatever pasta, walk, or library book is already in the week. It is not a 180-day plan. A short daily sit plus math noticed in the house is the shape the early-math panel still recommends, even though the evidence for minutes and for embedding in snack time is minimal — the studies bundled practices.[41] The honest pitch is *often enough that how-many is familiar*, not a dosage study.

Exact wording you can say

[40] Mary Budd Rowe, “Wait-Time and Rewards as Instructional Variables, Their Influence on Language, Logic, and Fate Control,” paper to the National Association for Research in Science Teaching, Chicago, April 1972, ERIC ED061103. Elementary science class; typical wait about one second after a question and 0.9 seconds after a student response; extending both to 3–5 seconds changes discourse. Not a homeschool trial and not a mathematics RCT. Robert J. Stahl, “Using Think-Time and Wait-Time Skillfully in the Classroom,” ERIC Digest ED370885, 1994. Think-time as a later instructional construct; about three seconds as a convention, not a sacrament; eight categories of silence; written for classroom teachers generally.

[41] Frye et al., *Teaching Math to Young Children* (2013), Recommendation 4 (encourage children to recognize and talk about math in everyday situations) and Recommendation 5 (dedicate time each day to teaching math and integrate math throughout the day). Both rated minimal evidence because studies bundled practices; the panel still recommends them. Not an isolated snack-time effect size.

On a pile:

"How many are there, really?"

"Did you count, or did you just see it?"

"Touch each. Say one number."

"The last number you said is how many. Try this pile."

On producing a set:

"Give me eight."

"Stop when you have eight. How do you know to stop?"

On compare:

"Which has more?"

"How could we check?"

When two counts disagree:

"You counted 8. I counted 7. Let's look."

When you are about to take over:

"Your hands. I'll wait."

Then wait.

Age-band moves: 5–7 / 8–10

5–7. Small collections. Subitize 1–3, then to 5. Count with one-to-one and cardinality to 10, then toward 20 and 30 as it holds. Produce a set of a given size. Compare two small piles. Write numerals beside the piles. A 0–10 or 0–20 line on the table or the floor: stand on 7, find which is closer to 10. Simple join and separate with objects — "we had 8, we used 2" — then a sketch, then a number sentence they can *see*. Speech is talk, not an essay. You write the caption if writing fights the sitting. Reciting to 100 may already be in the room. It is not the check.

8–10. Same moves, larger grain, if the small-set work held. Count and produce past 30, in different arrangements, keeping track. Compare without matching every pair, using the number. Locate two-digit numbers on a 0–100 line with endpoints. "About how many" as estimation — a handful, a parking row — then check by counting. Teen numbers as ten and some more sit in the next chapter; if they still treat 16 as 1 and 6, stay with how-many on teen piles this week and carry that miss forward. Write the numeral as a matter of course. A child of nine who recounts from one when asked "how many?" is still in the 5–7 band of this chapter, whatever the birthday. A child of six who can glance at four, count a scatter of twelve, produce eight, and place 7 on a line is not "too young" for the next idea.

Not a stable skill at either band: treating a chant to 100 as number sense; skipping the line because fractions have not arrived; withholding simple problems because "they're only counting." Same faculty. Different grain. Not two species of child.

First try-it for the student

A small pile already on the table, no vocabulary card:

eight pasta pieces, or whatever is actually there.

Say: "How many are there, really?"

Wait. If the child chants past the pile, or skips, they used recitation or lost one-to-one. Hear it. Ask them to touch each and say one number. If they finish and you ask "how many?" and they start over, they tagged and did not name the set. Say: "You already counted. The last number is how many." Hide the pile with a napkin and ask again. If they can say eight without restarting, cardinality is starting. Then: "Give me eight." If they cannot produce it, count out together from a larger pile and stop on purpose.

Later the same week, the diagnostic item:

the same question, on a pile of eight, after they have just recited to twenty or to one hundred for fun.

If they offer the recitation as the answer, the song is still wearing how-many clothes. Point back at the pile. The recitation can stay as a warm-up on another day. It is not this check.

How to fade help. First sitting: you count together, you say the last number is how many, they echo. Second: they count, you wait, you hint ("touch each"). Third: they count, say how many, and produce a set. Fourth: a cousin of the same job — stairs, a library picture, a glance of three — with the pasta in the cupboard. The word *cardinality*, if it arrives at all, arrives after the pile.

When to stop talking. When you hear yourself finishing the count. When the sit has become a lecture titled *What Number Sense Is*. When the child is mid-tag and you have already asked a second and a third question. One good question beats five. Stop while they still have a pile left in them. A second sitting later in the week beats a long sit that ends in a fight.

Practice that actually builds learning

Blocked, for a new move. The day you introduce "the last number is how many," give a short set that is only that: small piles in a line, count, ask how many, hide, ask again. The day you introduce producing a set, only "give me __." Mixing too early makes the child hunt for the "right" school move instead of looking at the objects.

Mixed, for when to use it. Later the same week, a glance of three next to a count of eight next to "which has more." Mixing is the practice of choosing: did I just see it, do I need to tag, do I compare? You impose that mix on the week you already have.

One incorrect example to diagnose. A recitation to one hundred offered as how-many. Or: they counted 8, you asked "how many?," they started at 1. Hear the song. Hear the restart. Neither is yet cardinality. The repair is a better question, then the pile, then — if needed — a named fact: "You already counted. Eight is how many." Silence in the face of a restart is not kindness.

These are illustrations, not reported families.

Named try-it: Kitchen count (in-home)

Time. Five to ten minutes. Then stop.

Materials. Pasta, eggs, crackers, spoons — whatever is already there. A scrap of paper for the numeral. A 0–20 number line of tape if you have one; a strip of paper if you do not.

Safety. Small pieces are a choking hazard for the youngest. Ordinary food hygiene. Let the child move the objects. A parent who feels rusty and takes the pasta can transmit tightness; the child's hands do the tagging.[42] The tone is curious, not courtroom. No shaming who ate what.

[42] Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L.

The fun. Making two bowls. Being the person who finds out how many.

The skill. Cardinality. Compare. A first join or separate: "we had 8 eggs, we used 2." How many are there, really? Recitation is not the check. Mind-change: recount.

Say: "How many are there, really?" If they chant, point at the bowl. If they skip, slow the touch. If they finish and restart on "how many?," name the last number as the pile. Then write 8. Then find 8 on the line. Kitchen motivates. The line and the numeral still get written.

Five-to-seven: piles to ten, in a line first. Eight-to-ten: a scatter, or two bowls to compare, or "about how many" before the count.

Named try-it: Subitize snack (in-home)

Time. Two to five minutes, when a snack is already happening.

Materials. One to five crackers, grapes, or apple slices. A napkin to cover and uncover.

Safety. Ordinary food. No using the glance as a trap about sneaking.

The fun. "Don't count — how many?" The tiny magic of seeing it.

The skill. IES Recommendation 1, step 1: small collections named without counting.

Uncover two, or three, or four. Wait. If they count, that is legal — then try a smaller set and ask them to see it. If they guess, reduce to two. Cover and ask again. Write the numeral once. A glance is not cheating. It is the first move.

Beilock, "Intergenerational Effects of Parents' Math Anxiety on Children's Math Achievement and Anxiety," *Psychological Science* 26, no. 9 (2015): 1480–1488. First- and second-graders of math-anxious parents learned less math over the year only when those parents reported frequent homework help; the association was math-specific; when such parents helped less often, it disappeared. School homework-help study, not a homeschool RCT. Used here as a caution to let the child move the objects.

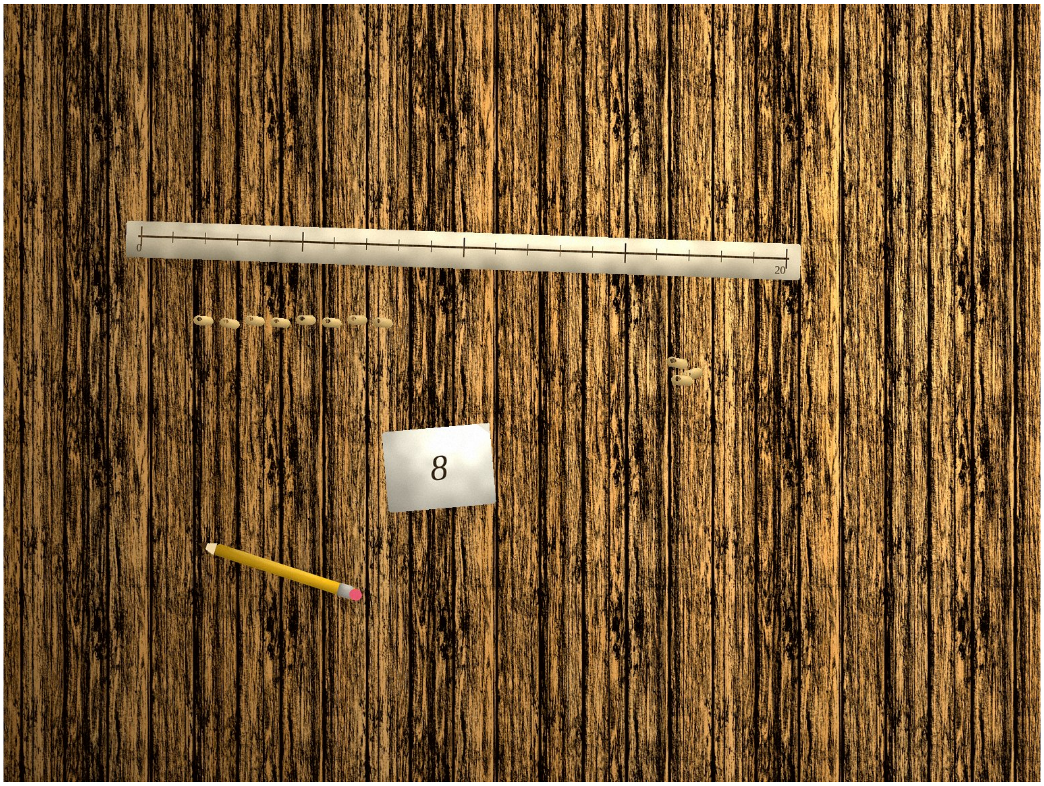

Figure 4: Pasta pieces in a neat row of eight next to a 0–20 number line of tape on a table; three pieces in a small glance-pile to the side; the numeral 8 written on a scrap. No people. No logos.

Named try-it: Walk that counts (out-of-home)

Time. Five to fifteen minutes of a walk you were taking anyway.

Materials. Stairs, fence posts, parked cars, cracks in a sidewalk. Optional: a 0–10 strip in a pocket for "where does 7 live?"

Safety. Roads. Water. Stay together. The child is the counter, not a lookout in traffic. Ordinary looking. No lecture to a stranger's child. No public quiz of a passerby.

The fun. Being the counter. Stairs as a vertical number line.

The skill. One-to-one in a line that is not a table row. Keeping track. Stairs and elevator buttons as numbers in space.

Say: “How many stairs to the landing, really?” Touch or step. One number each. At the top: “How many?” If they start over, they tagged the steps and did not name the flight. “You already counted. The last number is how many.” Elevator: the buttons are a line standing up. “We are on 2. Where is 5?” Five-to-seven: a short flight. Eight-to-ten: a longer flight, or “about how many parked cars on this side,” then check.

A walk is not a nature programme and not a trial of outdoor education. It is how-many in the world you were already in.

Named try-it: Library counting book and one question (out-of-home)

Time. Ten to twenty minutes.

Materials. Whatever counting book is on the shelf. Then, at home, a real pile.

Safety. Ordinary library. If a display is a live fight, walk to a different bay. Borrowing is the canon: a book that lets a child count, compare, or pattern, without a controversy as the content.

The fun. Choosing the book. Finding how-many in a picture, then how-many on the table.

The skill. How-many in the picture. Then how-many in the room, so the book is not the only set. Many homeschooling families already use the library as a source of books; a visit is ordinary, not a special programme.[43]

Say, on a picture of seven ducks: “How many are there, really?” Point. If they chant the rhyme and skip a duck, the rhyme was the recitation. Touch each. Then,

[43] National Center for Education Statistics, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement in Education Surveys* (NCES 2020-001), Table 6: 66 percent of homeschoolers used the library as a source of curriculum and books (kindergarten–grade 2, 78 percent; grades 3–5, 70 percent). NCES, *Parent and Family Involvement in Education: 2023* (NCES 2024-113), Table A-5: among the Homeschool-total row, 53 percent visited a library in the past month, versus 29 percent of all K–12. Different questions in different years; do not swap 53 for 66.

later: “There were seven in the book. Make seven with the pasta.” Picture, then objects, then numeral. The library hour is motivation. The pile and the line still happen.

Blocked / mixed, in this week. Monday: kitchen count, only cardinality. Tuesday: subitize snack, only the glance. Wednesday: mix glance and count. Thursday: walk that counts. Friday: library picture plus a home pile, and one diagnostic restart on “how many?”

Talk box

This is an illustration, not a reported family. A pile is on the table. The parent has not announced a definition.

Opening question

“How many are there, really?”

Not: “Can you count to one hundred?”

Follow-ups (pick three to five; you will not use all of them every sitting)

1. Did you count, or did you just see it?
2. Touch each. Say one number. Show me with things, then a sketch, then the number line, then a number.
3. Which has more? How do you know?
4. Does that answer make sense? You counted 8; I counted 7; let’s look.
5. You may change your mind. Try it another way. Recount.

How to wait

Ask. Count a slow three in your head. Look at the counters, the stairs, or the picture, not at the child’s face, if the silence is hard. After they stop talking, wait again before you speak. If they are mid-count or mid-reason, do not cut them off. A follow-up that arrives in half a second trains the child to wait *you* out, not to tag. Three seconds is a convention, not a sacrament — Stahl’s think-time, from classroom teaching. Rowe’s windows were science class. Use the pause. Neither is a homeschool trial of counting.

What a stuck silence usually means

They recited the sequence, and they can feel that the pile was not the job. Or wait-time after the question was zero, and they learned that you will fill it. Or they think the job is a performance of to-one-hundred. Or they recounted from one when asked "how many?" — no cardinality rule yet, a miss you can hear.[44] Or the question was vague. Or the numbers are too big. Or they are guessing what you want. Next move: smaller pile, wait, point back at the objects. "Touch each?" is a hint. The finished number is not a hint; it is the grab. Done enough this week: a spoken sentence. "There are eight. I counted. I can make eight."

For the student

This page is for you.

How many are there, really?

That is a question about a pile you can touch. It is not a song. A song that goes to one hundred is a good song. It is not how you know how many pasta pieces are on the plate.

You can sometimes just *see* how many. Two shoes. Three crackers. That glance is allowed. It is not cheating.

When there are more, you count. Touch each thing. Say one number. The last number you say is how many are there. You already counted. You do not have to start over when someone asks.

You can also *make* a pile. "Give me eight" means stop at eight. You know when to stop because eight is how many you want.

A tiny worked example

There are pasta pieces on the table.

You touch: one, two, three, four, five, six, seven, eight.

Someone asks, "How many?"

You say "eight." You do not start at one again.

[44] Frye et al., *Teaching Math to Young Children* (2013), Recommendation 1, including the parent-ear object that a child who recounts from one when asked "how many?" has not yet treated the last number as the cardinality of the set (see also the guide's Table 4 illustrations of counting errors).

Eight is the pile.

You can write 8. You can point to 8 on a number line. You can make another pile of eight.

If you and a grown-up get different numbers, you look again. You may change your mind. That is not losing.

Two tries

1. A real pile at home. Count. Say how many. Then hide the pile with a napkin and say how many again, without touching. Then make a pile of that size from a bigger bunch.

2. Stairs, or fence posts, or pictures in a library book. Count. Say how many, really. Then, if you can, find that number on a line or write it.

Write nothing if writing fights the sitting. Talk the number. Your grown-up can remember it. If you are older and writing is easy, you may write: "I counted **.** ***There are*** ."

Explain it back

Tell someone at the table, in your own words, how you know there are eight — or five, or three. Say whether you glanced or counted. Then make a pile of that size. Then say one time you started over when you did not need to, and what you can do instead.

Challenge

Someone says "I can count to one hundred," as if that answers how many crackers are on the plate. What did they use? How do you find out how many crackers there really are? Someone counts five, then starts at one when you ask how many. What did they forget? How would you check?

You are allowed to struggle. You may use pasta, stairs, a book, a number line. You count. You talk. If you get stuck, ask for a hint — not the finished number. Then try again.

When you talk, a sentence about the pile is enough for today. You do not have to write a paragraph. You do not have to call yourself a mathematician. Today you count, you say how many, and you can make that many.

A picture a computer made of pasta is a scene. It is interesting. It is not the pasta.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The child recites, and cannot yet answer "how many?" with the last number.

"I can count to one hundred." Or they finish a count and restart when you ask. Reciter is doing the work of Counter. Next move: smaller pile, in a line, touch each, then "you already counted — how many?" Hide the pile and ask again. Stay here if this is still the bottleneck after several weeks of short sits. Place value in the next chapter will not hold on top of a child who cannot name a set of eight. A human who will sit with a pile and wait, not a tablet that recites, is a reasonable next step if you have tried the small-set work and the restart is still the whole hour.

2. The child skips, double-tags, or cannot produce a set of a given size.

One-to-one is thin, or Producer has not joined Counter. Next move: objects that move. Slide each piece as it is tagged. A line, not a scatter. "Give me five" with a stop you model once, then they do it. Slow down the to-one-hundred talk until a set of five can be made and named. A child who can chant and cannot hand you seven is not ready to skip this. Go ahead once they can count a small line, say how many without restarting, and produce that many. A tutor is useful if tagging remains chaotic after a couple of weeks of daily object-first work *and* the hour has become a fight.

3. The child will not count aloud, or every sitting ends in a shrug, and the silence after your question is a wall.

Talk is part of the product, and a shrug is the mind at rest — or the wait was zero, or the question was "count to one hundred" in disguise, or they are guessing what you want. Next move: shorter sits, one pile, the opening question from the talk box, wait a slow three twice. If silence is hard, you look at the pasta, not at them. An authentic question is one you do not already have the numeral for. If you hear yourself finishing the count, you have started doing the work. If they offered a recitation, they may be stuck on the mismatch; name it and point at the pile. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that fills the silence, is the release valve.

When to slow down. Restarts still wearing how-many clothes. Skipping still the default. Sits so long that tagging never starts. A chant to 100 still offered as the whole of number. Those are brakes. “Not ready” because of age is the brake this book will not use. “Not ready” because a last number has not yet named a set is a real brake. A hundred chart is not a reason to skip eight pasta pieces.

When to go ahead. The child can count a small pile and say how many, not how far they can chant. They can produce a set of that size. A glance of one, two, or three has happened. A compare of two piles has happened, with a check. A numeral has been written beside a pile. A number line has held a whole number. Short sits are ordinary. Then place value in the next chapter has somewhere to sit. Being “good at counting” because a child loves the song to one hundred is not a reason to skip cardinality.

When to get a human tutor. You have run the pile, or the stairs, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear a recitation offered as how-many. Outsourcing the hearing is the thing to avoid, not asking for help.

Hearing which strand is thin — song without how-many, or how-many without a set they can make — is enough.

Correct a wrong count without crushing the attempt. “I hear one hundred. Let’s look at this pile.” Then tag together. “Never correct the child” is not this book’s rule. Hearing a wrong answer, asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The short version: the child talks and counts first. You hold the pile and the question. A tool may explain today’s idea *to you*, suggest extra piles you then vet, write a short script for *you* to say after the child has tried, offer a hint after an attempt, or help you diagnose a count the child already did. The child looks at objects. The child talks in the room.

Ages 5–7: no live open chat as the child’s partner. Ages 8–10: still parent in the room. Unaided first. A labelled SCRIPT after the try — a few lines for you to say, not a paragraph for the child to copy as their work.

Leave these out of the hour: a chatbot as the only partner; a tool that writes the child's count; a camera pointed at the page so a solution pops up; a detector score; a certificate of number sense; a tool asked to invent how many pasta pieces are in the bowl. Look. Ages five to ten are under 13. You hold the account.

A language model will happily finish the count. Treat every model-supplied "there are eight" as untrusted until it lives in the bowl. Fluent talk from a tool is performance, not the child's math. In a high-school math study, an unguarded chatbot made practice look better and left students worse when the window was closed. That paper is high-school mathematics, not a 5–10 trial. The crutch is still the sitting condition.[45] If the tool produces the number, the child is a spectator.

A pile, a number line, and a library book are tools too. Use them, then fade them.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 2 math workbook" is a publisher's scope, not a legal grade, and not a transcript line. A kitchen pile, a flight of stairs, and a library picture can place the same child in three different rooms, because they cut the grain differently. Choose by fit. This book names programmes as options in the resources chapter. It does not rank them. The record, when you need one, is a dated notebook, the task named, plus an exit ticket, plus one diagnostic item. The title a stranger can read is Mathematics, or Arithmetic. It is not Little Thinkers Math I.

Checklist before moving on

- You can hear a recitation to one hundred offered as how-many, and you can ask the cardinality question.
- You can tell Reciter from Corresponder from Counter in what the child *does*, without treating table ages as goals.

[45] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. GPT Base raised assisted practice grades 48 percent relative to control, then cut unaided exam grades 17 percent, in a high-school mathematics randomized trial in Turkey, nearly 1,000 students, grades 9–11. Not a 5–10 trial. The sitting condition — a tool that produces the response — is the mechanism used here.

- The child can name one, two, and three at a glance, at least some of the time.
- The child can count a small pile with one-to-one, say how many without restarting, and produce a pile of that size.
- At least one compare has happened: which has more, with a check.
- A numeral has been written beside a real pile.
- A whole number has lived on a number line — stand on 7, or point to 8 — not saved for fractions.
- At least one out-of-home count has happened: stairs, fence posts, or a library picture, then a home pile.
- Sits can be short. A spoken how-many is enough. You did not require a chant to 100 as the proof.
- You can hear a restart, a skip, and a song offered as a set, and you can ask a good question, without taking the pasta.

If most of that list is true, go on to place value, even if the birthday says otherwise. If the birthday says “second grade” and the last number still does not name the set, stay. The next chapter asks what the 1 means in 16. It needs a ten that is actually ten ones.

A path through elementary mathematics is the promise. A diploma is not. A percentile is not.

Chapter 2

Place value

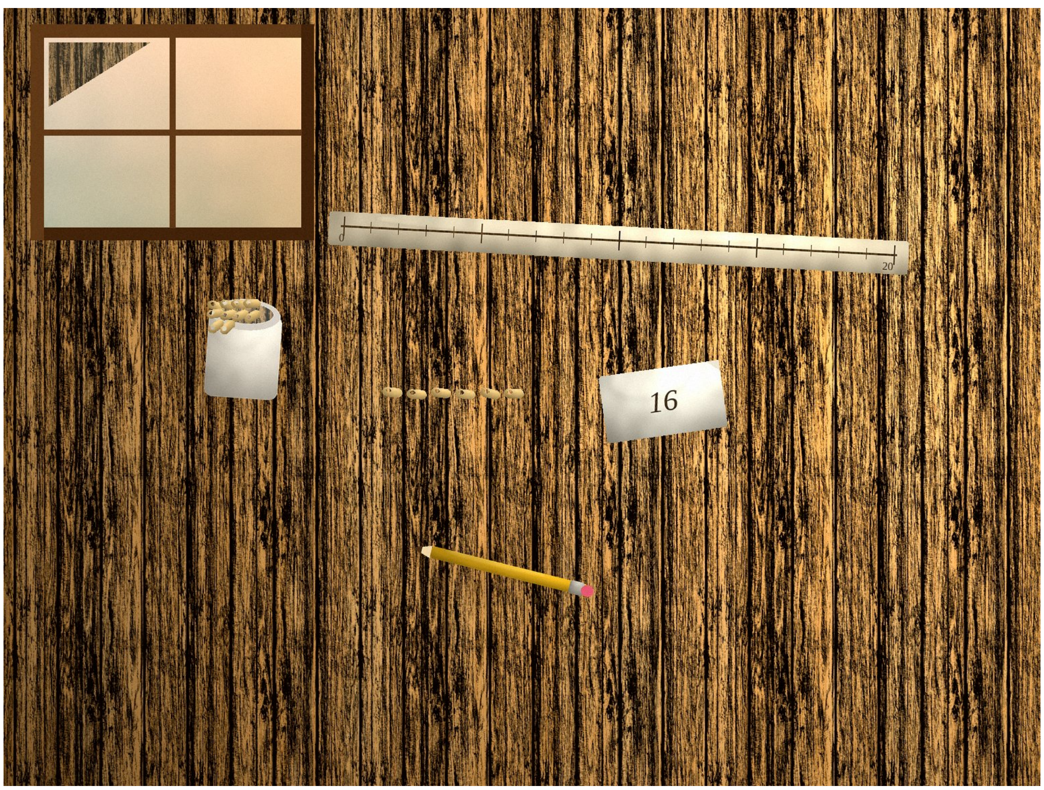

Figure 5: A kitchen-table still-life: ten pasta pieces in one cup beside six loose pieces, the numeral 16 written on a scrap, a 0–20 number line of tape. No people. No logos. No branded base-ten blocks.

Why this matters

What does the 1 mean in 16?

That is not a column-name question. It is not a test of whether the child can say *tens* and *ones*. It is a question about sixteen things that are actually on the table — pasta, sticks, pennies — and about whether the 1 is a ten, or just a digit sitting next to a 6.

A child can name the tens column and still think 16 is 1 and 6. They can recite a hundreds chart and still write 13 for thirty-one. They can add 27 + 15 as 3 and 12, or as 212, because the digits were treated as neighbors, not as tens and ones. Naming columns is vocabulary. Place value is base-ten structure.[46]

This chapter gives you one picture you can keep. Sixteen is one ten and six ones. The 1 is worth ten. Teen numbers as ten and some more are the foundation, not a vocabulary unit you finish before "real" place value begins. Ten ones *are* a ten. When you add 27 + 15, you are combining two tens and one ten, seven ones and five ones, and composing a new ten when the ones go past nine. The written method, when it arrives, is a record of that regrouping — not a stack of magic digits.

What this idea unlocks is the arithmetic of two-digit and three-digit numbers, and later the same system for larger numbers and, at the edge of this book, decimals. If "place value" still means a chart of column labels, later addition arrives as a trick in columns. If sixteen is a ten and six, 27 + 15 has somewhere to sit.

Two refusals belong in the same sitting as the picture, or the picture becomes a poster.

Place value is not knowing the ones-tens-hundreds chart. The National Mathematics Advisory Panel's whole-number foundation includes place value and the ability to compose and decompose whole numbers.[47] A robust sense of number, by the end of the elementary years, must include that structure. A chart can help

[46] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, March 2008), Critical Foundation 1 / Chapter 4: a robust sense of number includes understanding of place value and the ability to compose and decompose whole numbers. Naming columns is not that foundation. ERIC ED500486. Access date for URLs in these notes: 3 September 2026.

[47] National Mathematics Advisory Panel, *Foundations for Success* (2008), Critical Foundation 1, fluency with whole numbers, as in note 46.

you remember names. It does not show that ten ones match a ten.

Pictures forever, never writing 27 + 15, is the other miss. The Institute of Education Sciences 2021 intervention guide, Recommendation 3, strong evidence from 28 studies, wants a well-chosen set of concrete and semi-concrete representations, connected explicitly to notation, then faded.[48] When teaching place value, the representation for ones should be one-tenth the size of the representation for tens. If the representations were not proportional, place value would be harder to grasp.[49] Kitchen cups and pasta motivate. They do not replace the written 16, and they do not replace the number line.

This is a useful study, not a promise that every home will see the same result. The 2021 guide was written for struggling students in grades K–6, in small-group or one-on-one intervention. Steal the move. You are not running a school intervention hour by default.

The common U.S. school map, used here as a map and not as a homeschool statute, puts 11–19 as ten ones and some more ones in kindergarten; tens and ones in grade 1; numbers to 1,000 in grade 2, so that 853 is 8 hundreds, 5 tens, and 3 ones; later, to 1,000,000; and at the age-10 edge, decimals in the same place-value system.[50] Stay with tens and ones until they hold. Hundreds when they hold. Decimals as a label at the edge, not a middle-school finance course.

This book is not a reprint of *Mathematics for Homeschooling*. That book com-

[48]Lynn S. Fuchs, Nicole Bucka, Benjamin K. Frey, Barbara Foegen, Erica S. Lembke, and Russell Gersten, *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades* (WWC 2021006), Institute of Education Sciences, What Works Clearinghouse, 2021. Recommendation 3, strong evidence, 28 studies: use a well-chosen set of concrete and semi-concrete representations to support learning of mathematical concepts and procedures; connect them explicitly to notation; revisit; fade. Population: struggling students, grades K–6, small-group or one-on-one intervention. Steal the move; this is not a parent-hour RCT. This is a useful study, not a promise that every home will see the same result. https://ies.ed.gov/ncee/wwc/practiceguide/26

[49]Fuchs et al. (2021), Recommendation 3: "When appropriate, use representations that are proportional. For example, when teaching place value, the representation for ones should be one-tenth the size of the representation for tens." If the representations were not proportional, the concept of place value would be harder to grasp.

[50]National Governors Association Center for Best Practices and Council of Chief State School Officers, *Common Core State Standards for Mathematics* (2010). Kindergarten: work with numbers 11–19 to gain foundations for place value (ten ones and some more ones). Grade 1: tens and ones. Grade 2: multi-digit numbers to 1,000; 853 is 8 hundreds + 5 tens + 3 ones. Grade 4: to 1,000,000. Grade 5: decimals in the same place-value system. Used here as a map of typical U.S. public-school placement, not as a homeschool statute. http://www.corestandards.org/Math/

pressed elementary place value into a slice of two chapters. One pointer, then we teach 5–10 math at this grain. Counting in Chapter 1 is the prerequisite. If the last number still does not name a set of eight, stay there. This chapter needs a ten that is actually ten ones.

You do not need to be a mathematician. You do need to hear “the 1 is in the tens place” as vocabulary, and to ask what the 1 is worth, without taking the sticks.

This week you can learn to hear column names offered as place value, and to ask what the digit is worth. Today the child can show 16 as one ten and six ones, then write 16.

For the parent: understand it yourself

Many adults feel rusty on why the algorithm works. That is ordinary. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The child still makes the ten.

Everyday picture. Sixteen pasta pieces. Ten of them go in a cup. Six sit beside it. The cup is a ten. The six are ones. The numeral 16 is a record of that: one ten, six ones. If you dump the cup, you have sixteen ones again. Nothing magic happened. Ten ones *are* a ten.

A second kitchen picture, still ordinary. Sixteen cents as a dime and six pennies. The dime is a ten of pennies. Two ways to make 16 cents. Money motivates. You still write 16. A dime is not a lesson in decimals this week. It is a ten you can hold.

Precise picture. The 1 in 16 is worth ten, not one. The 6 is worth six. Sixteen is 10 + 6. It is one ten and six ones. It is not a 1 sitting next to a 6 the way a house number sits next to a street name.

Proportional representations make that visible. A ten-stick that is actually ten ones long, or a cup of ten pasta pieces next to one pasta piece, or a ten-frame you draw on paper with ten cells filled and six more beside it: the ten is ten times the one. A chip that “stands for” ten, the same size as a chip that stands for one, hides the idea. The 2021 guide’s rule is the one to steal: ones one-tenth the size of tens.[51]

[51] Fuchs et al. (2021), Recommendation 3, proportional-representation rule, as in note 49. Recommendation 6’s fluency examples include quickly evaluating place value to see whether regrouping is necessary.

Connect the objects to the numeral in the same visual field. Point at the cup, point at the 1 in 16. Point at the six pieces, point at the 6. Then write 16 = 10 + 6. Then find 16 on a 0–20 or 0–30 line. Objects, sketch, numeral, line. Then the objects fade so they do not become a crutch. Bring them back when a new size — hundreds, or a stubborn teen — needs a bridge.

Teen numbers are the foundation. Eleven is ten and one. Twelve is ten and two. Sixteen is ten and six. English hides this. *Sixteen* does not sound like *ten and six* the way some languages do. That is why you build it. A child who can chant "eleven, twelve, thirteen" and still writes 16 as 1 and 6 has the song, not the ten.

Two-digit numbers past the teens are the same idea, more tens. Twenty-seven is two tens and seven ones. It is 20 + 7. Thirty-one is three tens and one one. The reversal — 13 written for thirty-one — is a place-value miss, not messy handwriting. Hear it as "they heard *thirteen* as 1 then 3," or as "two-digit numerals are still concatenated digits." Show 31 as three cups and one piece. Write 31. Write 13 beside it and compare.

When you add 27 + 15, you combine tens with tens and ones with ones. Seven ones and five ones are twelve ones, which is one ten and two ones. Two tens and one ten and that new ten are four tens. 42. The algorithm, when you later write it in columns, is a record of that. Starting with the columns and adding pictures later trains the magic-digit reading. Starting with the cups and never writing 27 + 15 trains a toy. Sequence: situation → representation → equation → later, a generalizable written method the child can explain. This chapter owns the structure. Fluent multi-digit addition and subtraction sit across Chapters 3 and 4 as the 8–10 band, still with why-regrouping.

Estimation belongs beside exact tens. The Panel asked teachers to broaden computational estimation beyond rounding.[52] Age 5–7: "about how many." Age 8–10: "is 27 + 15 closer to 30 or to 50, and why." A ten-frame that is almost full is about ten. Twenty-seven is closer to 30 than to 20. That feel is place value, not a separate unit.

Hundreds, when tens hold: 853 is 8 hundreds, 5 tens, 3 ones. Ten tens *are* a hundred. Same proportional rule: the hundred is ten times the ten. Do not dump 1,000,000 this week. Do not dump decimals as the spine. At the 8–10 edge, a

[52] National Mathematics Advisory Panel, *Foundations for Success* (2008), Chapter 4 / estimation discussion: broaden instruction in computational estimation beyond rounding; the purpose of estimation is an appropriate approximation.

store tag such as $1.60 can raise a careful question: the 1 in $1.60 is not the 1 in 16. Naming that difference is decimal-adjacent, labelled, not a finance course.

A hundreds chart is a map of counting patterns. It is useful for +10 as a jump down a column, once tens are real. It is not place value by itself. A child can hop the chart and still think the 1 in 16 is a one.

Wrong answers you should be able to hear

1. *"The 1 is in the tens place."* Offered as the whole answer to "what does the 1 mean in 16?" Vocabulary without worth. Hear the column name. Ask what the 1 is *worth*. Show the ten.

2. *16 treated as 1 and 6, or as 7, or as 1 then 6 said as two numbers.* The digits are concatenated, not composed. Teen-as-ten-and-some-more is still the job. Bundle ten. Write 10 + 6. Write 16.

3. *13 written for thirty-one.* Tens and ones reversed. Hear *thirty-one* as three tens and one. Build 31 and 13 side by side. Find both on a line. The reversal is a place-value error, not a reason to wait a year for two-digit numbers.

4. *27 + 15 computed as 3 and 12, or as 212, or as 32.* These are different mistakes. The first adds digits without place. The second concatenates the sums (2+1=3, wait — 2+1=3 and 7+5=12, written 312, or 2 and 1 written beside 12 as 212). The third may be a ones error with the tens intact, or a tens error with the ones intact — ask which. None of them is "they cannot add." Show 27 with tens and ones, show 15, combine, then write.

5. *A chip, a mark, or a column header that "stands for ten" while looking the same as a one.* The representation is not proportional. Swap it for a cup of ten, a stick of ten, or a ten-frame that is actually ten cells. Then write the numeral.

A sixth you will also hear: they can bundle 16 and still freeze when you ask them to write 16 without the cup. The objects have not yet connected to notation, or they have become a crutch. Point at the cup and the 1 in the same look. Fade. Bring the cup back when 27 + 15 needs a bridge.

Five-minute parent warm-up

Do this before the lesson, with sixteen objects and a scrap of paper, no child in the room.

Minute 1. Bundle ten. Leave six. Say out loud: “Sixteen is one ten and six ones.” Write `16`. Write `10 + 6`. Circle the 1. Say what it is worth.

Minute 2. True or false, said aloud: the 1 in 16 is worth one. The 1 in 16 is worth ten. 16 is 1 and 6. 16 is ten and six. You are practicing the hearing, not racing.

Minute 3. Sketch two ten-sticks and seven small marks. Write 27, then 20 + 7, then “2 tens and 7 ones.” Sketch 15. Combine. Unpack ten ones from twelve ones. Write 42. That unpacking is what 27 + 15 needs.

Minute 4. Look at 16 and 61. Decide what a child who says “they both have a 1 and a 6” has used. Decide what “which is more, 16 or 61?” plus a build would do instead.

Minute 5. Write the sentence you will actually say: “What does the 1 mean in 16?” Put the pencil down. That sentence is the lesson.

If you can do those five minutes, you are ready to sit down. The child bundles. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version. Here is the shape scaled to this idea.

Warm-up (2–5 minutes, unaided). How-many on a small pile from Chapter 1. Then a glance: “is this about ten, or a lot less?” No device. No column chant as the warm-up.

Short model (3–7 minutes). Sixteen objects. You bundle ten. You name the ten and the ones. You write `16` and `10 + 6`. You show 16 on a line. You stop. You are showing what the 1 is worth, not delivering a lecture titled *Place Value*.

Student attempt (8–15 minutes). One to four numbers of *today’s size* — teens first, then other two-digit numbers. The child bundles. The child writes. You wait. Struggle before rescue: ask, wait, hint (“can you make a ten?”), then a short model on *your* set, not as their numeral.

One good question, then wait. “What does the 1 mean in 16?” After you ask, a slow three. After they stop, wait again. If they are mid-bundle, do not cut them off. Look at the cups, not at the child, if the silence is hard. Stahl's think-time, labelled classroom-origin; Rowe's windows were science class. Use the pause.[53]

Mixed practice (5–10 minutes). Yesterday's how-many next to today's teen. 16 beside 61. A write-it without the cup, then a check with the cup. Mixing is how the child learns *when* to bundle.

Exit ticket (2–4 minutes). Show 16 with tens and ones. Write 16. What is the 1 worth? One item from last week: how many in a small pile, without bundling. Done-enough is right, or wrong-with-a-reason we can use tomorrow.

Exact wording you can say

"What does the 1 mean in 16?"

"Show me with tens and ones."

"Can you make 16 another way?"

"If we add one more, what changes?"

"Is 16 closer to 10 or to 20?"

"Write it."

When they name the column:

"I hear *tens place*. What is the 1 worth?"

When 16 and 61 look "the same":

"Build both. Which has more?"

When you are about to take over:

"Your hands. I'll wait."

Age-band moves: 5–7 / 8–10

5–7. Teen numbers as ten and some more. Bundle ten ones. Ten-frames you draw. 11–19 built, written, found on a 0–20 line. Which is more, 16 or 61, with objects.

[53] Mary Budd Rowe, "Wait-Time and Rewards as Instructional Variables," NARST, April 1972, ERIC ED061103. Elementary science class; typical wait about one second after a question and 0.9 seconds after a student response. Not a mathematics RCT. Robert J. Stahl, "Using Think-Time and Wait-Time Skillfully in the Classroom," ERIC Digest ED370885, 1994. About three seconds as a convention; written for classroom teachers generally.

20, 21, 22 as two tens and some ones when teens hold. “About how many” for a handful. Money as a dime and pennies to make 16 cents, still writing 16. Simple 10 + 6 = 16 and 16 = 10 + 6, so the equal sign starts to see two names for the same amount — Chapter 3 will own that symbol on purpose. Speech is talk. You write if writing fights the sitting.

8–10. Two-digit tens and ones as a matter of course; hundreds when tens hold. 27 + 15 with tens and ones, then written as a record. 853 as 8 hundreds, 5 tens, 3 ones. “Is 27 + 15 closer to 30 or to 50, and why.” +10 and −10 as a jump of one ten. Bus numbers, parking stalls, “one more ten.” A store tag: what does this 1 mean, and is it the same as the 1 in 16? Label the decimal difference; do not dump a decimals course. Write the numeral as a matter of course. Fade the cups when they are accurate without them; bring them back for regrouping.

A child of nine who still treats 16 as 1 and 6 is still in the 5–7 band of this chapter, whatever the birthday. A child of six who can bundle 16, write it, and say the 1 is ten is not “too young” for 27 as two tens and seven ones.

First try-it for the student

Sixteen pasta pieces, or sixteen anything already in the house:

Say: “What does the 1 mean in 16?”

Wait. If they say “it’s in the tens place,” they used the column name. Hear it. Ask what the 1 is worth. If they say “one,” they read the digit. Bundle ten. Point at the cup and at the 1. If they say “ten,” ask them to show it and then write 16. Then: “Can you make 16 another way?” Ten and six, or sixteen ones, or a dime and six pennies later in the week.

Later the same week, the diagnostic item:

16 and 61, built side by side.

If they say the numbers are the same because both have a 1 and a 6, digits are still concatenated. Build. Compare. Find both on a line.

How to fade help. First sitting: you bundle, you name, they echo, you write together. Second: they bundle, you wait, you hint (“make a ten”). Third: they bundle, write 16, say what the 1 is worth. Fourth: write 16 without the cup, then check. The word *place value*, if it arrives at all, arrives after the ten.

When to stop talking. When you hear yourself filling the cup. When the sit has become a lecture titled *The Base-Ten System*. When the child is mid-bundle and you have already asked a second and a third question. One good question beats five. Stop while they still have a ten left in them.

Practice that actually builds learning

Blocked, for a new move. The day you introduce teen-as-ten-and-some-more, only teens. Bundle, write, say what the 1 is worth. The day you introduce two tens, only numbers in the twenties. Mixing too early makes the child hunt for a chart recitation.

Mixed, for when to use it. Later the same week, 16 next to 7 next to 61 next to “about how many.” Mixing is the practice of choosing: is this a ten-and-some-more job, a how-many job, a compare job?

One incorrect example to diagnose. “The 1 is in the tens place,” with no bundle and no worth. Or 16 built as one piece and six pieces. Hear the column. Hear the concatenated digits. The repair is a better question, then the cup of ten, then — if needed — a named fact: “The 1 is worth ten. Show me.” Silence in the face of a column name is not kindness.

These are illustrations, not reported families.

Named try-it: Bundle of ten (in-home)

Time. Ten to fifteen minutes. Then stop.

Materials. Sixteen pasta pieces, or beans, or counters. A cup, a paper ten-frame you draw, or a rubber band for a stick of ten. Scrap paper. A 0–20 number line of tape.

Safety. Small pieces are a choking hazard for the youngest. Ordinary table. Let the child make the ten.

The fun. Making a ten. Dumping it and making it again.

The skill. 16 is one ten and six ones. Then write 16. Then find 16 on the line. Kitchen motivates. The numeral and the line still happen.

Say: "What does the 1 mean in 16?" Bundle. Write `16` and `10 + 6`. Five-to-seven: teens, then maybe 20. Eight-to-ten: 27, 35, two tens and a leftover, then "add one more ten."

Named try-it: Money as tens (in-home)

Time. Eight to twelve minutes.

Materials. Dimes and pennies, or ten pennies you trade for a dime. Scrap paper.

Safety. Coins are a choking hazard for the youngest. Count them back into a dish. No tasting. No using the coins as a sermon about money or about a food system. A dime is a ten of pennies this week.

The fun. Making 16 cents two ways.

The skill. A dime is a ten of pennies — still write 16. Money motivates. It does not replace the written number, and it does not replace the pasta ten from the same week.

Say: "Show me 16 cents with a ten and some ones." Then: "Show me another way." Write 16. If they say the dime is "one," they read the digit on the coin as worth. Compare a dime next to ten pennies. Same amount. The 1 in 16 is still a ten.

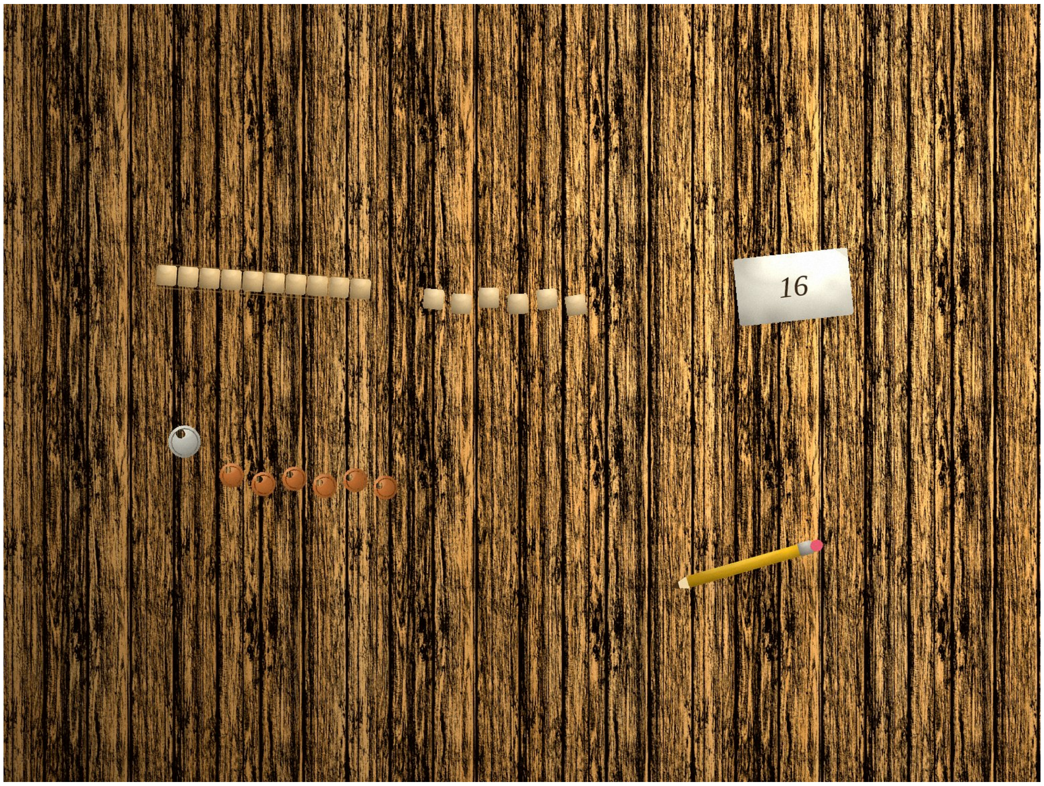

Figure 6: A drawn ten-stick (ten connected squares) beside six small squares; the numeral 16; a dime and six pennies in the background, not instead of the ten-stick. No logos. No people.

Named try-it: Bus and parking numbers (out-of-home)

Time. Five to ten minutes of a trip you were taking anyway.

Materials. A posted bus number, a parking stall, a house number you can see from the sidewalk. Optional: a scrap for 16 and 61.

Safety. Ordinary looking. Roads. No interrogation of a driver. No stepping into a stall. No lecture to a stranger.

The fun. “One more ten.” Reading a number that was already in the world.

The skill. Place value in the wild. Five-to-seven: which is more, 16 or 61? Eight-to-ten: what does this 2 mean in 27? What would one more ten be?

Say: "What does the 1 mean in that 16?" If there is no 16, pick the number that is there: "What does the 2 mean in 23?" Build it later at home with cups. The parking lot is motivation. The cups and the written numeral still happen at the table.

Named try-it: Store tag (out-of-home)

Time. Three to eight minutes.

Materials. A generic price tag on a shelf. Scrap if you want to copy the digits.

Safety. Aisle courtesy. Hands to yourself. A price is a number. It is not a sermon about a food system. No shoplifting a "test." No public quiz of a cashier.

The fun. What does the 1 mean?

The skill. Five-to-seven: a whole-dollar price, or a tag you copy as 16 and ask about 16. Eight-to-ten edge: the 1 in $1.60 is not the 1 in 16. Naming that difference is decimal-adjacent, labelled. You still write the number. You do not dump hundredths as a course.

Say: "What does this 1 mean?" If they say "one dollar" for $1.60, that is a fair reading of money as measurement of value — Chapter 7 will own coins and prices as measurement. This week, compare it to the 1 in 16 on your scrap. Different worth. Same digit. That surprise is the 8–10 edge.

Blocked / mixed, in this week. Monday: bundle of ten, only 16. Tuesday: other teens. Wednesday: mix 16, 7, and 61. Thursday: bus or parking number. Friday: money as tens, plus one diagnostic column-name.

Talk box

This is an illustration, not a reported family. Sixteen objects are on the table. The parent has not announced a chart.

Opening question

“What does the 1 mean in 16?”

Not: “Which column is the tens column?”

Follow-ups (pick three to five; you will not use all of them every sitting)

1. Show me with tens and ones. What is the 1 worth?
2. Can you make 16 another way? Show me with things, a sketch, the number line, a number sentence.
3. If we add one more, what changes? Is 16 closer to 10 or to 20?
4. Does that answer make sense? Build 16 and 61. Which has more?
5. You may change your mind. Try it another way. Write it.

How to wait

Ask. Count a slow three in your head. Look at the cups, the dime, or the tag, not at the child’s face, if the silence is hard. After they stop talking, wait again. If they are mid-bundle, do not cut them off. Three seconds is a convention, not a sacrament — Stahl’s think-time. Rowe’s windows were science class. Use the pause. Neither is a homeschool trial of place value.

What a stuck silence usually means

They named the column, and they can feel that worth was asked for. Or they think 16 is 1 and 6. Or the numbers are too big. Or they offered a chart recitation. Or wait-time after the question was zero. Or they are guessing what you want. Next move: smaller number (16, not 853), wait, point back at the ten. “Can you make a ten?” is a hint. “It’s worth ten” said by you is not a hint; it is the grab. Done enough this week: a spoken sentence. “The 1 means ten. Sixteen is one ten and six ones.”

For the student

This page is for you.

What does the 1 mean in 16?

It does not mean one. It means ten.

Sixteen is one ten and six ones. You can put ten pasta pieces in a cup and leave six beside it. You can write `16`. You can write `10 + 6`. Those are the same amount.

A song that names "tens place" is a useful song. It is not what the 1 is worth. Show the ten.

A tiny worked example

Sixteen pieces on the table.

You make a ten. Six are left.

The 1 in 16 is the ten. The 6 is the six ones.

You write `16`.

If someone says 16 and 61 are the same because both have a 1 and a 6, you can build both. 61 is six tens and one one. It is more. You can find both on a number line.

You may change your mind when you look. That is not losing.

Two tries

1. Make 16 with a ten and some ones. Write 16. Say what the 1 means. Then dump the ten and make 16 another way.
2. A number in the world: a bus, a parking stall, a price tag you copy onto paper. What does the first digit mean? Build that number at home.

Write nothing if writing fights the sitting. Talk the ten. If you are older and writing is easy, you may write: "16 is 1 ten and 6 ones. The 1 is worth ___."

Explain it back

Tell someone at the table, in your own words, what the 1 means in 16. Show it with things. Show it on a line. Then say what would happen if you added one more ten.

Challenge

Someone says "the 1 is in the tens place," and does not show a ten. What did they use? How do you show what the 1 is worth? Someone writes 13 for thirty-one. What happened? How would you build 31 and 13 to check?

You are allowed to struggle. You may use cups, a dime, a line, a scrap of paper. You bundle. You talk. If you get stuck, ask for a hint — not the finished numeral. Then try again.

When you talk, a sentence about the ten is enough for today. You do not have to write a paragraph. You do not have to call yourself a mathematician. Today you show 16 as one ten and six ones, and you write 16.

A picture a computer made of blocks is a scene. It is interesting. It is not the ten on your table.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The child names the column, and cannot show what the digit is worth.

"Tens place" without a ten. Vocabulary is doing the work of place value. Next move: sixteen objects, one cup, write 16 next to the cup. Ask what the 1 is worth. Stay here if this is still the bottleneck after several weeks of short sits. Addition in the next chapter will not hold on top of concatenated digits. A human who will sit with cups and wait, not a chart that recites, is a reasonable next step if you have tried the teen work and the column name is still the whole hour.

2. The child treats 16 as 1 and 6, or reverses 13 and 31, or adds 27 + 15 as concatenated digits.

The digits are neighbors, not tens and ones. Next move: smaller numbers, proportional tens, two numbers built side by side. 16 and 61. 13 and 31. Combine 10 + 6 before 27 + 15. Slow down the hundreds chart until a teen is a ten and some more. A child who can hop a chart and cannot bundle 16 is not ready to skip this. Go ahead once they can show 16, write it, and say the 1 is ten. A tutor is useful if reversals remain the default after a couple of weeks of daily bundling *and* the hour has become a fight.

3. The child will not bundle, or every sitting ends in a shrug, and the silence after your question is a wall.

The wait was zero, or the question was a chart recitation in disguise, or the numbers are too big, or they are guessing what you want. Next move: shorter sits, only 16, the opening question from the talk box, wait a slow three twice. If silence is hard, you look at the cup, not at them. If you hear yourself filling the cup, you have started doing the work. If they offered a column name, name the miss and point at the ten. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that fills the silence, is the release valve.

When to slow down. Column names still wearing worth-clothes. 16 still 1 and 6. Cups still a toy, never connected to 16. Sits so long that bundling never starts. Those are brakes. “Not ready” because of age is the brake this book will not use. “Not ready” because a ten is not yet ten ones is a real brake. A standard algorithm worksheet is not a reason to skip the cup.

When to go ahead. The child can show 16 as one ten and six ones and write 16. They can say what the 1 is worth. 16 and 61 have been built and compared. A number in the world has been read for worth, then built at home. A line has held 16. Short sits are ordinary. Then addition, subtraction, and the equal sign have somewhere to sit. Being “good at place value” because a child can recite ones-tens-hundreds is not a reason to skip the ten.

When to get a human tutor. You have run the cup, or the dime, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear a column name offered as worth. Outsourcing the hearing is the thing to avoid, not asking for help.

Correct a wrong worth without crushing the attempt. “I hear tens place. Let’s see what the 1 is worth.” Then bundle. Hearing a wrong answer, asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The child bundles and talks first. You hold the sixteen and the question. A tool may explain today’s idea *to you* from this chapter, suggest extra teen numbers you then vet, write a short script for *you* to say after the child has tried, offer a hint after an attempt, or help you diagnose a numeral the child already wrote. Crop to the paper. Do not upload the child’s face.

Ages 5–7: no live open chat as the child’s partner. Ages 8–10: still parent in the room. Unaided first. SCRIPT after the try.

Leave these out of the hour: a chatbot as the only partner; a tool that writes 16 for them; a camera pointed at a worksheet so a solution pops up; a detector score; a certificate of place value; a tool asked to invent how many pieces are in the cup. Look.

A language model will happily name the tens column. Treat every model-supplied "the 1 is in the tens place" as untrusted until a ten sits on the table. In a high-school math study, an unguarded chatbot made practice look better and left students worse when the window was closed. That paper is high-school mathematics, not a 5–10 trial.[54] If the tool produces the numeral, the child is a spectator.

Cups, a dime, a ten-frame you draw, and a number line are tools too. Use them, then fade them.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 2 math workbook" is a publisher's scope, not a legal grade, and not a transcript line. The record, when you need one, is a dated notebook, the task named, plus an exit ticket, plus one diagnostic item. The title a stranger can read is Mathematics, or Arithmetic. It is not Little Thinkers Math I, and it is not a publisher's level name.

Checklist before moving on

- You can hear "the 1 is in the tens place" as vocabulary, and you can ask what the 1 is worth.
- The child can show 16 as one ten and six ones, then write 16.
- Teen numbers are ten and some more, not 1 and 6.
- At least one compare of 16 and 61, or 13 and 31, has happened with objects and a line.
- A proportional ten has been used: a cup of ten, a stick of ten, or a ten-frame, not a same-size chip that "stands for" ten.

[54] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. GPT Base +48% on assisted practice, −17% on the unaided exam, high-school mathematics, Turkey, nearly 1,000 students, grades 9–11. Not a 5–10 trial. The sitting condition — a tool that produces the response — is the mechanism used here.

- The numeral was written in the same sitting as the bundle. The objects are starting to fade.
- At least one out-of-home number has been read for worth: bus, parking, or a store tag, then built at home.
- 8–10, as it holds: 27 + 15 with tens and ones, or 853 named as hundreds, tens, and ones; “closer to 30 or to 50” with a reason.
- Sits can be short. A spoken worth is enough. You did not require a hundreds-chart recitation as the proof.
- You can hear concatenated digits, a reversal, and a column name, and you can ask a good question, without filling the cup.

If most of that list is true, go on to addition, subtraction, and the equal sign, even if the birthday says otherwise. If the birthday says “third grade” and 16 is still 1 and 6, stay. The next chapter asks what the equal sign means, and how you know 8 is the same as 3 + 5. It needs amounts that are actually amounts.

A path through elementary mathematics is the promise. A diploma is not. A percentile is not.

Chapter 3

Addition, subtraction, and the equal sign

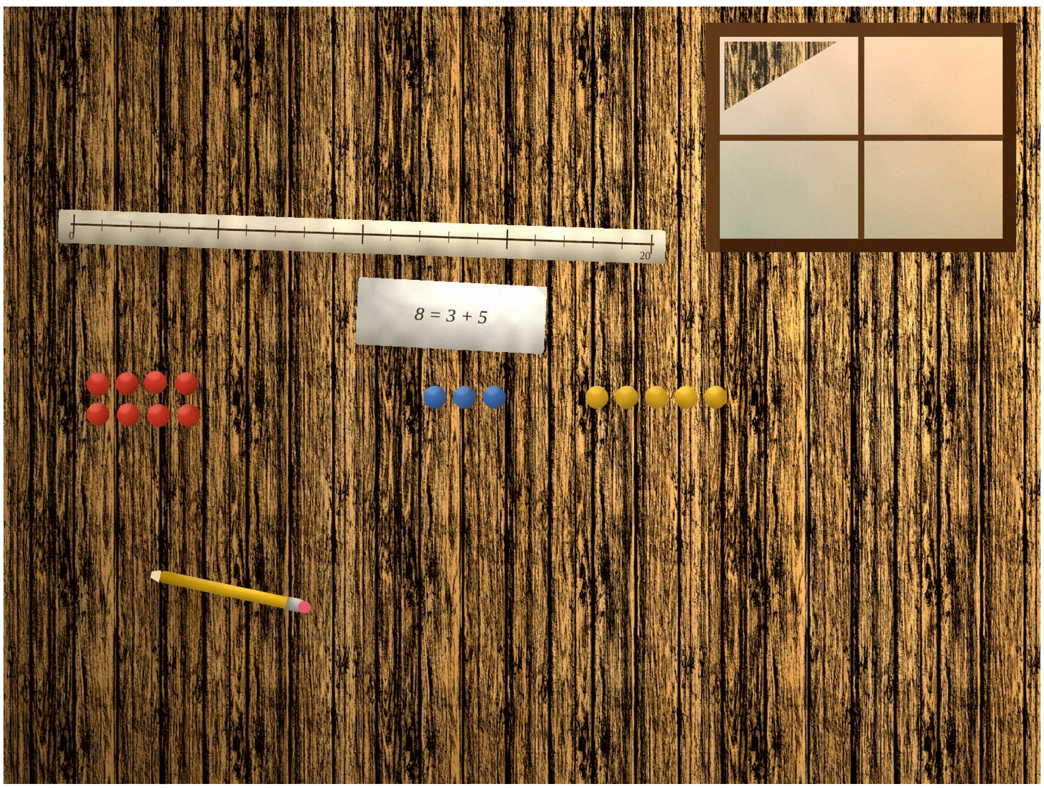

Figure 7: A kitchen-table still-life: eight counters on the left of a scrap that reads 8 = 3 + 5; a group of three and a group of five on the right; a 0–20 number line of tape. No people. No logos. No branded balance toy.

Why this matters

What does the equal sign mean here?

How do you know 8 is the same as 3 + 5?

Those are not prealgebra questions. They are not a test of whether the child can say *equals*. They are questions about two amounts that are actually on the table — eight on one side, three and five on the other — and about whether = says "the same as," or "here comes the answer."

A child can compute 3 + 5 and still freeze at `8 = 3 + 5`. They can write 12 in the box for `8 + 4 = ☐ + 5`, because they added everything to the left of the equal sign. They can write 17, because they added every number on the page. Both answers treat = as a green light to compute, not as a relation between two sides.[55]

This chapter gives you one meaning you can keep. The equal sign means "the same as." Eight is the same amount as three and five. You write `8 = 3 + 5` in the 5–7 band, not as a unit you save for later. You also write `3 + 5 = 8`. You write `3 + 5 = 5 + 3`. You write `6 + 4 = 7 + 3`. You mark true and false: `5 + 2 = 8 - 1` is true; `8 + 4 = 12 + 5` is false. The symbol the child is already seeing has a meaning. Teach the meaning.

What this idea unlocks is later algebra, and also this week's arithmetic. Virtually all work on equations needs = as a relation.[56] If the symbol still means "the answer comes next," later pages arrive as a hunt for the box. If both sides name the same amount, join, separate, put-together, take-apart, and compare have somewhere to sit — and so does `8 + 4 = ☐ + 5`.

[55] Eric J. Knuth, Ana C. Stephens, Nicole M. McNeil, and Martha W. Alibali, "Does Understanding the Equal Sign Matter? Evidence from Solving Equations," *Journal for Research in Mathematics Education* 37, no. 4 (2006): 297–312. Many elementary and middle-school students view the equal sign as "an announcement of the result of an arithmetic operation rather than as a symbol of mathematical equivalence." They report Karen P. Falkner, Linda Levi, and Thomas P. Carpenter (1999): many first- through sixth-graders solving `8 + 4 = ☐ + 5` answered 12, 17, or 12 and 17. Falkner 1999 was not independently opened for this book; it is reported as Knuth reports it. Access date for URLs in these notes: 3 September 2026.

[56] Knuth et al. (2006), citing Thomas P. Carpenter, Megan Loef Franke, and Linda Levi, *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School* (Portsmouth, NH: Heinemann, 2003): a limited conception of the equal sign is "one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding that the equal sign represents a relation."

Why teach this *now*, at five to ten, not in a prealgebra chapter? Eric Knuth, Ana Stephens, Nicole McNeil, and Martha Alibali asked middle-schoolers — older than this book's ceiling — for their best definition of the equal sign. Relational ("the same as") was 32 percent of sixth graders, 43 percent of seventh, and 31 percent of eighth, in a sample of 177 students at one school. There was no linear improvement by grade.[57] Many students still saw = as announcing a result. Relational understanding predicted correct equation solving even after controlling for mathematics scores, in a subsample.[58] The equal sign is traditionally introduced in the early elementary years, with little instructional time spent on it later. Curricular materials, they report, show relational uses less often than operational uses. Waiting until middle school is how the operational view gets taught. That is why `8 = 3 + 5` belongs on this week's scrap.

This is a useful study, not a promise that every home will see the same result, and not this child's score. Knuth's percents describe older students in a school sample. Steal the implication: write numbers on both sides of = while the symbol is still new.

Addition and subtraction in this chapter are actions and relations, not keyword lists. Something joins. Something separates. Two parts make a whole. Two amounts are compared. The unknown can be the result, the change, or the start. "Altogether" is a situation word you have to interpret, not a cue to add. Pictures, then equations. Pictures forever, never writing `8 = 3 + 5`, is a miss. Equations first, never a pile, is the other miss. Kindergarten on the common U.S. map already encourages `5 + 2 = 7`.[59]

This book is not a reprint of *Mathematics for Homeschooling*. That book com-

[57] Knuth et al. (2006). Sample: 177 students, one ethnically diverse Midwest middle school. Best definition coded relational ("the same as") versus operational. Relational: 32% of sixth graders (n = 47), 43% of seventh (n = 72), 31% of eighth (n = 58). No significant linear or quadratic grade trend. Knuth's sample is grades 6–8, older than this book's ceiling — which is why `8 = 3 + 5` is taught now. These percents are not this child's score.

[58] Knuth et al. (2006). Relational understanding predicted correct equation solving after controlling for standardized mathematics scores (subsample N = 65). This is a useful study, not a promise that every home will see the same result.

[59] National Governors Association Center for Best Practices and Council of Chief State School Officers, *Common Core State Standards for Mathematics* (2010), Kindergarten overview: equations such as `5 + 2 = 7` and `7 — 2 = 5`; student writing of equations in kindergarten is encouraged but not required. Grade 1 critical area: add-to, take-from, put-together, take-apart, and compare; adding two is the same as counting on two; making tens; relationship between addition and subtraction. Used here as a map, not a homeschool statute. http://www.corestandards.org/Math/

pressed elementary operations into a slice of two chapters. One pointer, then we teach 5–10 math at this grain. Counting and a ten that is actually ten ones are the prerequisites. If 16 is still 1 and 6, stay in Chapter 2. This chapter needs amounts that are actually amounts.

You do not need to be a mathematician. You do need to hear 12 for `8 + 4 = □ + 5` as an operational equal sign, not a cute slip, and to ask what = means here, without taking the pencil.

This week you can learn to hear "the answer comes next," and to put numbers on both sides. Today the child can write `8 = 3 + 5` and say "the same as."

For the parent: understand it yourself

Many adults feel rusty on why `8 = 3 + 5` is legal. That is ordinary. A diet of `3 + 5 = 8` trains the other reading: operations on the left, a blank on the right, = meaning "the total." Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The child still builds both sides.

Everyday picture. Eight grapes on a plate. You can see them as five and three, or as four and four, or as seven and one. The plate is still eight. Nothing "comes next." The amount is the amount, named two ways. `8 = 5 + 3` is a record of that look. `5 + 3 = 8` is the same relation written the other way.

A second kitchen picture, still ordinary. Three crackers, then five more join them. How many now? That is join, result unknown. You had eight. Some were eaten. Five are left. How many were eaten? That is separate, change unknown. Maya has eight. Jordan has three more. How many does Jordan have? That is compare, not "altogether." The cover story can hide the structure. The type is the structure.

Precise picture. Write `8 = 8`. Write `8 = 3 + 5`. Write `3 + 5 = 8`. Write `3 + 5 = 5 + 3`. Write `6 + 4 = 7 + 3`. Circle every equal sign. They all mean the same thing: both sides name the same amount.

`8 + 4 = □ + 5` is the diagnostic you should be able to hear. Falkner, Levi, and Carpenter, as Knuth reports them, found many first- through sixth-graders answering 12, 17, or both.[60] Twelve: add the left, put the total in the box. Seventeen: add

[60] Falkner, Levi, and Carpenter (1999), as reported in Knuth et al. (2006). McNeil and Alibali (2005) operational definitions ("the total," "the answer") and McNeil et al. (2011) `4 = 2 + 2` formats are the same research line; page-level experimental numbers from McNeil 2011 were not

every number on the page. Both: the child is running two operational readings at once. The box is 7, because both sides are 12. Put 8 + 4 on one side of the table and □ + 5 on the other. Build them. The equal sign is the same-as between those two sides.

True/false equations train the same hearing. `7 = 7` is true. `4 + 3 = 5 + 2` is true. `4 + 3 = 8` is false. `5 + 2 = 8 − 1` is true. The child has to look at both sides. A diet of only fill-in-the-blank on the right trains the operational view.

Addition and subtraction are types of action and relation. The names this book uses, from the common school map and from the 2021 intervention guide, are enough for a parent hour.[61]

Join (change plus / add-to). A start amount, something added, a result. Unknown can be result, change, or start. "We had 8. 3 more came. How many now?" Result unknown. "We had 8. Some more came. Now we have 13. How many came?" Change unknown. "Some were on the plate. 3 joined. Now there are 8. How many were there at first?" Start unknown.

Separate (change minus / take-from). A start amount, something removed, a result. Unknown in any slot. "We had 8. We used 3. How many now?" Result. "We had 8. We used some. 5 are left. How many did we use?" Change. Start-unknown separate is the hard cousin: "Some were there. We used 3. 5 are left. How many were there at first?"

Put-together / take-apart (combine / part–part–whole). Two parts make a whole. No action required. "3 red, 5 green, how many in all?" Whole unknown. "8 in all, 3 red, how many green?" Part unknown. This is a natural home for `8 = 3 + 5`.

Compare. Two amounts and a difference. "Maya has 8. Jordan has 5. How many more does Maya have?" Difference unknown. Compared-quantity unknown and referent unknown are the harder cousins. "How many more" is not a keyword for subtract. It is a compare story. Match, or line, or subtract, depending on what is

independently opened here.

[61] CCSS-M Grade 1, as in note 59. Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades* (WWC 2021006), 2021, Recommendation 5, strong evidence, 18 studies: teach students to identify word-problem types that include the same type of action or event. Named types: Change (join or separate), Combine, Compare. Identifying a type is different from determining the operation; it is not useful to associate a type with an operation. Population: struggling K–6, intervention. Steal the moves. Table 5.2 of that guide is the keyword trap: *altogether*, *more*, *left*, *each* fail as operation cues.

unknown.

Identifying a type is different from picking an operation. It is not useful to glue a type to "always add" or "always subtract."[62] Vary the unknown. Vary the cover story. A first-grade teacher study of children's problem types found that teachers who taught from types taught problem solving more and number facts less — and their students exceeded controls on facts, problem solving, reported understanding, and confidence.[63] Facts were not sacrificed. Exact classroom cell means sit in the notes. The portable finding is the one to steal: types and facts belong together.

Pictures then equations. Objects, a sketch, a number line, a number sentence. Point at all four. Then fade the objects so they do not become a crutch. Kindergarten writing of equations is encouraged, not required. In this house, write `8 = 3 + 5` when both sides are visible. An 8–10 child writes it as a matter of course, including `8 + 4 = □ + 5` and `6 + 4 = 7 + 3`.

The number line is already in the room from Chapters 1 and 2. Join can be a hop forward. Separate can be a hop back. Compare can be two positions and the distance between them. Kitchen grapes motivate. They do not replace the line, and they do not replace the written equation.

Wrong answers you should be able to hear

1. *12 for `8 + 4 = □ + 5`.* The child added everything to the left. The equal sign was "the answer comes next." The box was "where the answer goes," even though a 5 was already sitting on the right. Put both sides on the table.
2. *17 for the same item, or 12 and 17 both.* The child added every number, or ran both operational readings. Same miss, bigger grab. Build 8 + 4. Build □ + 5. What would make the sides the same?
3. *A stare at `8 = 3 + 5`, or "you wrote it backwards."* The only legal sentence

[62] Fuchs et al. (2021), Recommendation 5, definition box, as in note 61.

[63] Thomas P. Carpenter, Elizabeth Fennema, Penelope L. Peterson, Chi-Pang Chiang, and Megan Loef, "Using Knowledge of Children's Mathematics Thinking in Classroom Teaching: An Experimental Study," *American Educational Research Journal* 26, no. 4 (1989): 499–531. 20 first-grade teachers randomised to a month-long workshop on children's addition/subtraction problem solving; 20 control. Experimental teachers taught problem solving more and number facts less; students exceeded controls on number facts, problem solving, reported understanding, and confidence. Full PDF cell means and student N were not copied beyond 20/20 teachers (ERIC ED292683). The portable finding: teaching from children's problem types moved both problem solving and facts. Facts were not sacrificed.

they have seen is `3 + 5 = 8`. The operational diet did the teaching. Write `8 = 8` first. Then `8 = 3 + 5` with objects under both sides.

4. *"Altogether" as add, on a story that is equal groups or compare.* A keyword grab. Four cartons of twelve is not 4 + 12. "How many more" is not always subtract. Ask what kind of story. Show it. Chapter 8 will lock the types as a durable classification; this chapter owns join, separate, put-together, take-apart, and compare.

5. *Every 8 + 5 still starts at 1, recounting both sets, after they can already count on.* Count-all is a real early strategy. Once they can count, composing and counting on from the larger is the next move. Retrieval of known facts is Chapter 4. This week, meaning first.

A sixth you will also hear: they can build 3 + 5 = 8 and still refuse `8 = 3 + 5` as "not a real problem." The format is the diet. Mix the formats on purpose.

Five-minute parent warm-up

Do this before the lesson, on a scrap of paper, no child in the room.

Minute 1. Write `8 = 3 + 5`. Say out loud: "Eight is the same amount as three and five." Write `8 = 4 + 4`. Write `5 + 3 = 8`. Write `3 + 5 = 2 + 6`. Circle the equal signs.

Minute 2. True or false, said aloud: `7 = 7`. `4 + 3 = 5 + 2`. `4 + 3 = 8`. `5 + 2 = 8 - 1`. You are practicing the hearing, not racing.

Minute 3. Look at `8 + 4 = □ + 5`. Decide, without filling the box, what 12 would mean and what 17 would mean. The box is 7, because both sides are 12. You are installing the two errors so you can hear them tomorrow.

Minute 4. Sketch a join (8, then 3 more), a separate (8, 3 used), and a compare (8 and 5, how many more). One equation for each, including one with the total on the left.

Minute 5. Write the sentence you will actually say: "What does the equal sign mean here?" Under it: "How do you know 8 is the same as 3 + 5?" Put the pencil down. Those sentences are the lesson.

If you can do those five minutes, you are ready to sit down. The child builds both sides. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version. Here is the shape scaled to this idea.

Warm-up (2–5 minutes, unaided). Two facts the child already knows, untimed. One non-canonical equation, such as `6 = 4 + □`. Paper. No device.

Short model (3–7 minutes). One new idea, one picture, one written sentence. Eight objects. Split into 3 and 5. Write `8 = 3 + 5`. Show both sides. You talk for a few minutes. Then you stop.

Student attempt (8–15 minutes). One to four items of *today's type* — true/false, or join result-unknown, or `8 = 3 + 5` shape. The child builds. The child writes. You wait. Struggle before rescue: ask, wait, hint ("show both sides"), then a short model on *your* scrap, not as their equation.

One good question, then wait. "What does the equal sign mean here?" or "How do you know 8 is the same as 3 + 5?" A slow three. After they stop, wait again. Look at the two sides, not at the child, if the silence is hard. Stahl's think-time; Rowe's windows were science class.[64]

Mixed practice (5–10 minutes). Yesterday's how-many or teen-as-ten next to today's equation. A compare story next to a number bond with the total on the left. Mixing is how the child learns *when* = is a relation, not a green light.

Exit ticket (2–4 minutes). `8 = 3 + 5` true or false. One join or separate. One item from last week. Done-enough is right, or wrong-with-a-reason we can use tomorrow.

Exact wording you can say

"What does the equal sign mean here?"

"How do you know 8 is the same as 3 + 5?"

"Can you show both sides?"

"Is 3 + 5 the same as 5 + 3?"

[64] Mary Budd Rowe, "Wait-Time and Rewards as Instructional Variables," NARST, April 1972, ERIC ED061103. Elementary science class. Not a mathematics RCT. Robert J. Stahl, "Using Think-Time and Wait-Time Skillfully in the Classroom," ERIC Digest ED370885, 1994. About three seconds as a convention; classroom teachers generally.

"Try `6 + 4 = 7 + 3`."

"What would make this false?"

When they write 12 in the box:

"You wrote 12. Let's put 8 + 4 on this side and □ + 5 on that side."

When they glue a word to an operation:

"What kind of story is this — something joined, something compared, two parts?"

When you are about to take over:

"Your hands. I'll wait."

Age-band moves: 5–7 / 8–10

5–7. Small numbers. `8 = 8`. `8 = 3 + 5`. `3 + 5 = 8`. `3 + 5 = 5 + 3`. True/false with totals under 10, then 20. Join and separate with result unknown, then change unknown. Put-together as `8 = 3 + 5` with two colors. Compare-difference-unknown with matching or a line. Objects, sketch, line, equation. You write if writing fights the sitting. Count-on from the larger when count-all is clean. Start-unknown can wait until result and change are easy, or it can appear with tiny numbers and objects.

8–10. Same meaning, larger grain. `8 + 4 = □ + 5` as a regular item, not a trick. `6 + 4 = 7 + 3` and `25 + 8 = 23 + □`. True/false with subtraction on one side. Join, separate, combine, compare with the unknown in different slots, including start-unknown. Two-digit addition and subtraction as a record of tens and ones from Chapter 2, still with why-regrouping. A strip or two bars for compare. Write as a matter of course. Fade objects when both sides are visible in the numerals; bring them back when 12 appears in the box.

A child of nine who writes 12 for `8 + 4 = □ + 5` is still in the 5–7 band of this chapter for the equal sign, whatever the birthday. A child of six who can write `8 = 3 + 5` and say "the same as" is not "too young" for two names of eight.[65]

First try-it for the student

[65] National Mathematics Advisory Panel, *Foundations for Success* (2008), Chapter 5: many U.S. middle-school students do not understand the concept of mathematical equality. Finding 15: "too young" claims wrong if prerequisites are present. Finding 10: conceptual understanding, computational fluency, and problem solving belong together.

Eight objects, already on the table:

Split them into 3 and 5, or let the child split.

Say: "How do you know 8 is the same as 3 + 5?"

Wait. If they say "because 3 + 5 is 8," they used the operational sentence, which is true and incomplete. Write `8 = 3 + 5`. Ask what = means here. If they say "the answer comes next," they used the green light. Put eight on the left and 3 + 5 on the right. Same pile, two names. If they say "the same as," ask them to show both sides and then try `6 + 4 = 7 + 3`.

Later the same week, the diagnostic item:

`8 + 4 = □ + 5`

If they write 12 or 17, the operational view is still in the room. Build both sides. Do not scold. The box is 7.

How to fade help. First sitting: you split, you write `8 = 3 + 5`, they echo "the same as." Second: they split, you wait, you hint ("show both sides"). Third: they write `8 = 3 + 5` and a true/false. Fourth: `8 + 4 = □ + 5` with objects, then without. The word *relational*, if it arrives at all, arrives after the two sides.

When to stop talking. When you hear yourself filling the box. When the sit has become a lecture titled *The Meaning of Equality*. When the child is mid-build and you have already asked a second and a third question. One good question beats five. Stop while they still have an equation left in them.

Practice that actually builds learning

Blocked, for a new move. The day you introduce `8 = 3 + 5`, only totals on the left, small numbers, objects visible. The day you introduce true/false, only true/false. Mixing too early makes the child hunt for "where the answer goes."

Mixed, for when to use it. Later the same week, `8 = 3 + 5` next to a join story next to `8 + 4 = □ + 5`. Mixing is the practice of choosing: same amount, or a join, or a missing addend?

One incorrect example to diagnose. 12 for `8 + 4 = □ + 5`. Hear the operational equal sign. The repair is a better question, then both sides on the table, then

— if needed — a named fact: “Equals means the same as. What would make these sides the same?” Silence in the face of 12 is not kindness.

These are illustrations, not reported families.

Named try-it: True/false equations (in-home)

Time. Eight to twelve minutes. Then stop.

Materials. Paper, counters, a 0–20 number line.

Safety. Ordinary table.

The fun. Catching a false one. Being the person who knows why.

The skill. `8 = 3 + 5` is true. `5 + 2 = 8 − 1` is true. `8 + 4 = □ + 5` is not 12. Five-to-seven: small numbers. Eight-to-ten: larger, still relational.

Say: “What does the equal sign mean here?” Offer three: `8 = 3 + 5` (true), `4 + 3 = 8` (false), `6 + 4 = 7 + 3` (true). Build both sides of each. Write true or false. Then one box: `8 + 4 = □ + 5`. Kitchen table. The equation still gets written. The line still holds 8 and 12.

Named try-it: Join and separate with a snack (in-home)

Time. Five to ten minutes, when a snack is already happening.

Materials. The actual snack. Scrap for the equation.

Safety. Food hygiene. No shaming who ate what. No using the story as a trap about sneaking. The tone is curious, not courtroom.

The fun. How many now?

The skill. Vary the unknown: result, change, start. Write the equation including the `8 = 3 + 5` shape, not only `3 + 5 = 8`. Kitchen motivates. The equation still happens.

Say: “We had 8. 3 joined. How many now?” Write `8 + 3 = 11` and also `11 = 8 + 3`. Later: “We had 8. Some were eaten. 5 are left. How many were eaten?” `8 −`

$\square = 5$ or $8 = 5 + \square$. Five-to-seven: result unknown, then change. Eight-to-ten: start unknown, still with the snack in sight.

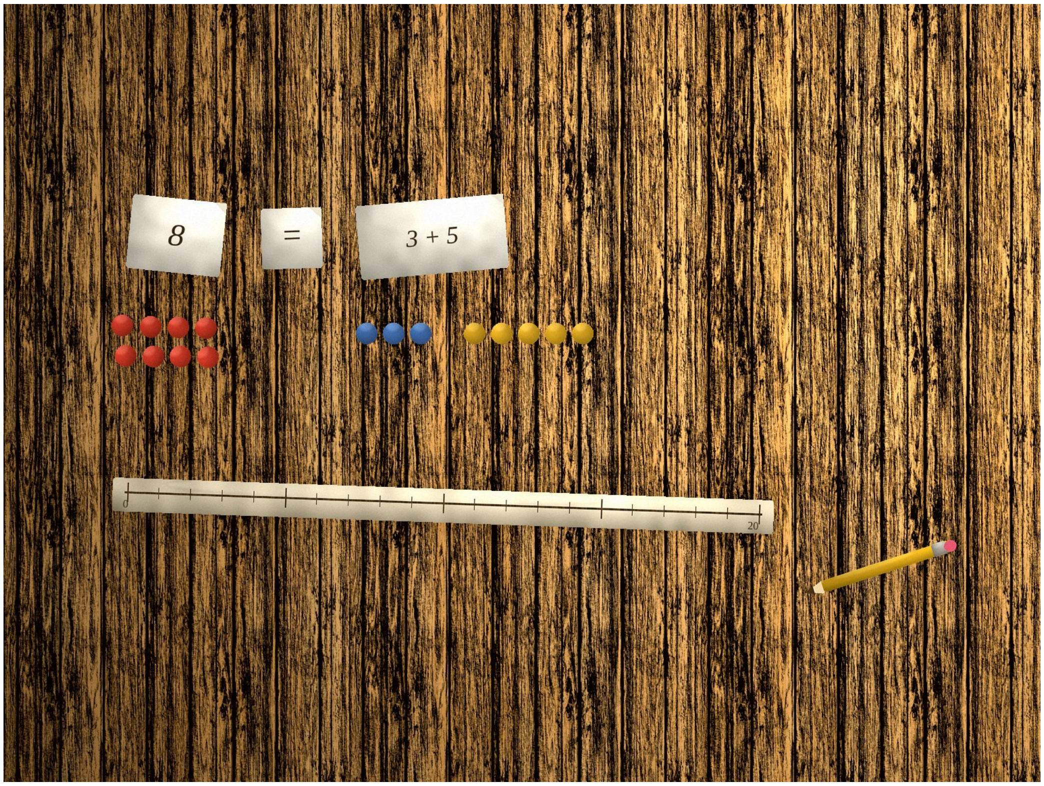

Figure 8: Two sides of a written sentence: 8 on the left of an equal sign, 3 + 5 on the right; counters under each side; a number line in the same field. No people. No logos.

Named try-it: Compare on a walk (out-of-home)

Time. Five to ten minutes of a walk you were taking anyway.

Materials. Two piles of pinecones, two parked bikes, two windows, two short walls. A scrap if you want the equation later.

Safety. Ordinary looking. Roads. No taking what is not yours. No lecture to other families. No public quiz.

The fun. Which is more, how many more.

The skill. Compare-difference-unknown, not "altogether." Then write the equation at home, or on the scrap. The walk motivates. The equation still happens.

Say: "Which has more? How many more?" Match, or count both, or look. "How many more" is the difference, not a keyword. Five-to-seven: small visible sets. Eight-to-ten: larger sets, or "Jordan has 3 more than Maya; Maya has 8; how many does Jordan have?" as compared-quantity unknown, still built with a sketch when you sit.

Named try-it: Store, how many more to make a pack (out-of-home)

Time. Three to eight minutes.

Materials. A price, a pack count on a generic box, coins if you already have them.

Safety. Aisle courtesy. A price is a number. It is not a sermon about a food system. No shoplifting a "test." No public quiz of a cashier. Coins stayed a choking hazard at home beforehand.

The fun. Making the amount. How many more to make a dozen, a pack, a dollar.

The skill. Separate-change-unknown / missing addend. Still write the equation. `8 + ☐ = 12`. `12 = 8 + ☐`. Five-to-seven: small whole numbers. Eight-to-ten: making a dollar as 100 cents, still a missing addend, labelled, not a decimals course.

Say: "We have 8. The pack is 12. How many more to make the pack?" If they add 8 + 12 because "more" appeared, they grabbed a keyword. Ask what kind of story. Show 8, show 12, show the gap. Write it when you sit, if the aisle is not the place.

Blocked / mixed, in this week. Monday: `8 = 3 + 5` with objects. Tuesday: true/false only. Wednesday: join/separate snack, mix the equation shapes. Thursday: compare on a walk. Friday: `8 + 4 = ☐ + 5` as the diagnostic, plus one store missing-addend.

Talk box

This is an illustration, not a reported family. Eight objects are on the table. `8 = 3 + 5` is written, or about to be. The parent has not announced a definition.

Opening question

“What does the equal sign mean here?”

or

“How do you know 8 is the same as 3 + 5?”

Not: “What is the answer?”

Follow-ups (pick three to five; you will not use all of them every sitting)

1. Can you show both sides? What does = say?
2. Show me with things, a sketch, the number line, a number sentence. Is 3 + 5 the same as 5 + 3?
3. Try `6 + 4 = 7 + 3`. What kind of story is this — joined, separated, two parts, compared?
4. Does that answer make sense? You wrote 12 in the box; let’s put 8 + 4 on this side and □ + 5 on that side.
5. You may change your mind. What would make this false? Try it another way.

How to wait

Ask. Count a slow three in your head. Look at the two sides, not at the child’s face, if the silence is hard. After they stop talking, wait again. If they are mid-reason, do not cut them off. Three seconds is a convention, not a sacrament — Stahl’s think-time. Rowe’s windows were science class. Use the pause. Neither is a homeschool trial of the equal sign.

What a stuck silence usually means

They think = means “the answer comes next,” and they can feel that both sides were asked for. Or they added all the numbers. Or they offered a keyword. Or wait-time after the question was zero. Or the numbers are too big. Or they are guessing what you want. Next move: smaller numbers, both sides built, wait, point back at the objects. “Show both sides?” is a hint. “The box is 7” said by you is not a hint; it is the grab. Done enough this week: a spoken sentence. “Equals means the same as. 8 is the same as 3 + 5.”

For the student

This page is for you.

What does the equal sign mean here?

It means the same as.

How do you know 8 is the same as 3 + 5?

You can show eight. You can show three and five. Same amount. Two names.

`8 = 3 + 5` is true. `3 + 5 = 8` is true too. `3 + 5 = 5 + 3` is true. The equal sign does not mean “here comes the answer.” It means both sides name the same amount.

A tiny worked example

Eight counters.

Three on this napkin. Five on that napkin.

You write `8 = 3 + 5`.

Someone gives you `8 + 4 = ☐ + 5`.

If you write 12, you added the left and parked the total in the box. The 5 on the right was already there.

Put 8 + 4 on one side. That is 12.

Put ☐ + 5 on the other side. That also has to be 12.

The box is 7.

You may change your mind when you look. That is not losing.

Two tries

1. Make 8 two ways. Write `8 = ___ + ___`. Write a true one and a false one. Say why.
2. A real join or separate: a snack, or two piles on a walk. Write the equation two ways if you can — total on the right, and total on the left.

Write nothing if writing fights the sitting. Talk “the same as.” If you are older and writing is easy, you may write: “8 is the same as 3 + 5 because ___.”

Explain it back

Tell someone at the table, in your own words, what the equal sign means. Show 8 as 3 + 5. Then say what 12 would mean on `8 + 4 = □ + 5`, and what you would do instead.

Challenge

Someone always puts the total on the right, and says `8 = 3 + 5` is backwards. What did they use? How do you show both sides? Someone sees “more” in a story and always subtracts. How do you ask what kind of story it is — joined, separated, two parts, or compared?

You are allowed to struggle. You may use counters, a snack, a line, a scrap of paper. You build. You talk. If you get stuck, ask for a hint — not the finished number. Then try again.

When you talk, a sentence about the same amount is enough for today. You do not have to write a paragraph. You do not have to call yourself an algebra student. Today you write `8 = 3 + 5` and say “the same as.”

A picture a computer made of a balance is a scene. It is interesting. It is not the eight on your table.

If it isn’t clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The child treats = as “the answer comes next,” and writes 12 (or 17) for `8 + 4 = □ + 5`.

The operational view is doing the work of the relation. Next move: both sides built with objects. `8 = 8` first. Then `8 = 3 + 5`. Then true/false. Then the box, with 8 + 4 on one napkin and □ + 5 on the other. Stay here if this is still the bottleneck after several weeks of short sits. Fluency in the next chapter will not hold on top of a green-light equal sign; known facts retrieved into the wrong meaning stay the wrong meaning. A human who will sit with two sides and wait, not a worksheet of `3 + 5 = □` only, is a reasonable next step if you have tried the true/false work and 12 is still the whole hour.

2. The child can compute 3 + 5 and cannot yet join, separate, or compare a story, or glues a keyword to an operation.

"Altogether" as add. "Left" as subtract. The cover story is doing the work of the type. Next move: one type, objects, unknown in the result slot, then the change slot. Ask what happened. Draw it. Then write. Slow down the keyword hunt until a join can be built. A child who can fill blanks and cannot say whether something was added or compared is not ready to skip this. Go ahead once they can write `8 = 3 + 5`, mark a true/false, and act out one join and one separate. A tutor is useful if keyword grabs remain the default after a couple of weeks of daily type work *and* the hour has become a fight.

3. The child will not write an equation, or every sitting ends in a shrug, and the silence after your question is a wall.

The wait was zero, or the only legal sentence they know is `3 + 5 = 8`, or the numbers are too big, or they are guessing what you want. Next move: shorter sits, eight objects, the opening question from the talk box, wait a slow three twice. If silence is hard, you look at the two sides, not at them. If you hear yourself filling the box, you have started doing the work. If they offered 12, name the miss and build both sides. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that fills the silence, is the release valve.

When to slow down. 12 still in the box. `8 = 3 + 5` still "backwards." Keywords still the default. Sits so long that building never starts. Those are brakes. "Not ready" because of age is the brake this book will not use. "Not ready" because both sides are not yet the same amount is a real brake. A prealgebra workbook is not a reason to skip eight grapes.

When to go ahead. The child can write `8 = 3 + 5` and say "the same as." A true/false has been marked with both sides shown. 12 for the box has been heard and rebuilt. A join and a separate have been acted and written. A compare has had "how many more" without a keyword sermon. Short sits are ordinary. Then fluency of known facts has somewhere to sit. Being "good at addition" because a child is fast on `3 + 5 = □` is not a reason to skip the other side of the equal sign.

When to get a human tutor. You have run the two sides, or the snack, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear 12 as an operational equal sign. Outsourcing the hearing is the thing to avoid, not asking for help.

Correct 12 without crushing the attempt. "I hear 12. Let's put 8 + 4 here and box-plus-five there." Then look. Hearing a wrong answer, asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The child builds both sides first. You hold the equation and the question. A tool may explain today's idea *to you* from this chapter, suggest extra true/false items you then vet, write a short script for *you* to say after the child has tried, offer a hint after an attempt, or help you diagnose an equation the child already wrote. Crop to the paper. Do not upload the child's face. The child never sees the answer key.

Ages 5–7: no live open chat as the child's partner. Ages 8–10: still parent in the room. Unaided first. SCRIPT after the try.

Leave these out of the hour: a chatbot as the only partner; a tool that fills the box; a camera pointed at the page so a solution pops up; a detector score; a certificate of algebra-readiness; a tool asked to invent how many crackers were eaten. Look.

A language model will happily write 12. Treat every model-supplied box as untrusted until both sides sit on the table. In a high-school math study, an unguarded chatbot made practice look better and left students worse when the window was closed. That paper is high-school mathematics, not a 5–10 trial.[66] If the tool produces the 7, the child is a spectator.

[66] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. GPT Base +48% on assisted practice, −17% on the unaided exam, high-school mathematics, Turkey, nearly 1,000 students, grades 9–11. Not a 5–10 trial. The sitting condition — a tool that produces the response — is the mechanism used here.

Counters, a snack, a number line, and a scrap with `8 = 3 + 5` are tools too. Use them, then fade them.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 2 math workbook" is a publisher's scope, not a legal grade, and not a transcript line. The record, when you need one, is a dated notebook, the task named, plus an exit ticket, plus one diagnostic item. The title a stranger can read is Mathematics, or Arithmetic. It is not Little Thinkers Math I.

Checklist before moving on

- You can hear 12 for `8 + 4 = □ + 5` as an operational equal sign, and you can put both sides on the table.
- The child can write `8 = 3 + 5` and say "the same as."
- True/false equations have been marked with both sides shown.
- A join and a separate have been acted, then written, including at least one equation with the total on the left.
- A compare has happened: which is more, how many more, without gluing "more" to an operation.
- At least one out-of-home missing-addend or compare has happened: walk, store pack, or "how many more to make," then an equation at the table.
- Objects, a sketch, a number line, and a number sentence have appeared in the same week. Kitchen did not replace the line or the equation.
- 8–10, as it holds: the box item is regular; start-unknown has appeared with objects; two-digit add/sub still names tens and ones.
- Sits can be short. A spoken "the same as" is enough. You did not require a prealgebra unit as the proof.
- You can hear 12, 17, "backwards," and a keyword, and you can ask a good question, without filling the box.

If most of that list is true, go on to fluency, even if the birthday says otherwise. If the birthday says "third grade" and 12 is still in the box, stay. The next chapter asks which facts the child already knows for sure. It needs meaning first. Timing facts the child cannot yet get right is a different miss.

A path through elementary mathematics is the promise. A diploma is not. A percentile is not.

Chapter 4

Fluency

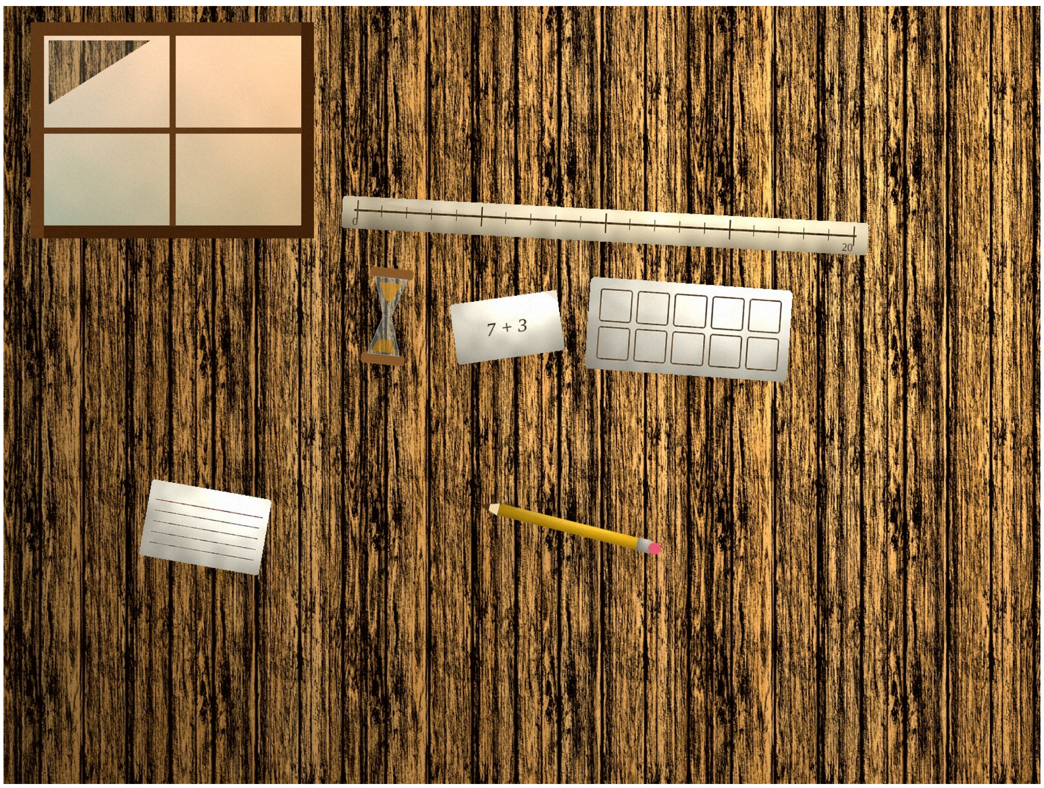

Figure 9: A kitchen-table still-life: a one-minute sand timer beside a tiny home-made card that reads 7 + 3, a ten-frame of paper with ten cells, a 0–20 number line of tape. No people. No logos. No branded flashcard deck. No leaderboard.

Why this matters

Which of these do you already know for sure?

That is not a race question. It is not a test of whether the child is "good at math." It is a question about a small set of facts on the table — 7 + 3, 8 + 2, 5 + 5 — and about whether those facts are already right when nobody is timing.

A child can be fast and wrong. A child can be slow and right, and then, later, faster on the ones that were already right. A kitchen timer used to rank a seven-year-old is not fluency. A timed worksheet of facts the child cannot yet get right untimed is not fluency. A franchise of flashcards sold as the first teaching move is not fluency.[67]

This chapter gives you one order you can keep. Accurate first. Then, sometimes, timed for one to five minutes, only on known facts. The Institute of Education Sciences 2021 intervention guide, Recommendation 6, strong evidence from 27 studies, wants brief timed activities as *one way* to build fluency — after the child has been working on the idea over many lessons, not as the lesson, not as mere timed worksheets, and not as an unsupported computer program.[68] The National Mathematics Advisory Panel's Finding 11 wants automatic recall of addition and related subtraction facts, and later of multiplication and related division facts, because fast, accurate, effortless processing frees working memory for the next step.[69] Those

[67]Lynn S. Fuchs, Nicole Bucka, Benjamin K. Frey, Barbara Foegen, Erica S. Lembke, and Russell Gersten, *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades* (WWC 2021006), Institute of Education Sciences, What Works Clearinghouse, 2021. Recommendation 6, strong evidence, 27 studies (21 without WWC reservations, 6 with): "Regularly include timed activities as one way to build fluency in mathematics." Population: struggling students, grades K–6, small-group or one-on-one intervention. Steal the constraints; this is not a parent-hour RCT and not a timed-test kit. This is a useful study, not a promise that every home will see the same result. Isolation of timing as the active ingredient is an open question; the 27 studies bundled timing with other intervention components. Access date for URLs in these notes: 3 September 2026. https://ies.ed.gov/ncee/wwc/practiceguide/26

[68]Fuchs et al. (2021), Recommendation 6, how-to constraints, copied: timed activities last 1 to 5 minutes and are not the entire focus of the intervention; they are one component of a multi-component intervention; add timed activities once students have been working on a concept over many lessons; do not use timed activities to introduce and teach mathematics concepts and operations; identify already-learned topics; the panel does not recommend merely giving students timed worksheets or putting students on a computer-based program without supporting their learning.

[69]National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, March 2008), Finding 11: computational proficiency depends on sufficient and appropriate practice to develop

two sentences belong together. Practice is required. Timing is a way to practice retrieval of what is already known. Timing is not how a child learns what 8 + 7 means.

What this idea unlocks is everything that uses a fact as a stepping-stone: two-digit addition that needs 7 + 5 inside it, equal groups that need 3×4 inside them, a compare story that needs $12 - 8$. If fluency still means a race, later pages arrive as panic. If fluency means known facts you can get without rebuilding every time, the next problem has room in working memory.

Two refusals belong in the same sitting as the order, or the order becomes a poster.

Timed tests do not create fluency. The 2021 panel's own constraints, copied: timed activities last 1 to 5 minutes; they are not the entire focus; add them once students have been working on a concept over many lessons; do not use them to introduce and teach concepts and operations; identify already-learned topics; the panel does not recommend merely giving timed worksheets or putting students on a computer program without supporting their learning.[70] Steal those constraints. This is a useful study, not a promise that every home will see the same result. The 27 studies were run as part of multi-component interventions for struggling students in grades K–6. Timing was bundled with other support. Isolation of timing as the single active ingredient is an open question. Steal the move: a short retrieval of facts the child already knows.

Speed as identity is the talent story the Panel wants unwound. Finding 14: changing children's beliefs from a focus on ability to a focus on effort increases engagement and mathematics outcomes. Public resignation about mathematics is rooted in the idea that success is largely a matter of inherent talent, not effort.[71] A parent

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automatic recall of addition and related subtraction facts, and of multiplication and related division facts, fluency with standard algorithms, and understanding of commutative, associative, and distributive properties. Chapter 5, general principles of learning: automaticity is "fast, accurate, and effortless processing of content information—which frees up working memory for more complex aspects of problem solving." ERIC ED500486.

[70] Fuchs et al. (2021), Recommendation 6, as in notes 67–2. Russell Gersten et al., *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools* (NCEE 2009-4060), Recommendation 6, moderate evidence: about 10 minutes per intervention session on fluent retrieval — an intervention-session figure, not a parent-hour law.

[71] National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 14: changing children's beliefs from a focus on ability to a focus on effort increases engagement and mathematics outcomes. Public resignation about mathematics "seems rooted in the idea that success in mathematics is largely a matter of inherent talent, not effort."

who treats the timer as proof of being a "math person" is teaching the talent story. Retrieve known facts. Leave the public race out of the identity of being good at math.

This book is not a timed-test kit. It is not a flashcard franchise. It is not a reprint of *Mathematics for Homeschooling*, which compressed elementary fluency into a slice of two chapters. One pointer, then we teach 5–10 math at this grain. Meaning from Chapters 1–3 is the prerequisite. If 16 is still 1 and 6, or if 12 still appears in the box for `8 + 4 = □ + 5`, stay there. This chapter times only what is already right.

The 5–7 object is addition and related subtraction within 10, then 20, once the child can count, compose, and show the fact with objects. Doubles, making 10, count-on. Untimed accuracy first. A one-minute known-fact retrieval later. The 8–10 object is multiplication and related division facts, once arrays and equal groups are real — Chapter 5 owns the groups; this chapter owns retrieval of the facts that are already right on paper. Multi-digit procedures are a different fluency and still want conceptual talk: why this regrouping works, not forty of the same item.

You do not need to be a mathematician. You do need to hear a fast wrong answer as timing-too-early, and to delay the clock, without taking the pencil.

This week you can learn to hear speed offered as identity, and to ask which facts are already known for sure. Today the child can pick three facts they know for sure and say them.

For the parent: understand it yourself

Many adults remember fluency as a stack of timed sheets. That is ordinary, and it is not this chapter's object. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The child still says the facts.

Everyday picture. 7 + 3. The child can build it with a ten-frame: seven, three more, ten. The child can write `7 + 3 = 10` and `10 = 7 + 3`. Ask, with no clock: "What is 7 + 3?" If they say 10, without rebuilding, that fact is a candidate for later retrieval. If they still count all from one, it is not yet a known fact. Practice the meaning. The clock waits.

A second kitchen picture, still ordinary. A one-minute sand timer. Three facts on scraps: 5 + 5, 8 + 2, 6 + 4. The child already gets these right untimed. One minute, just these. How many did they get right? Tomorrow, the same three, maybe one more that became known this week. Beating yesterday's self, if a simple tally helps. Not beating a sibling. Not a leaderboard. Not fifty mixed facts they do not know.

Precise picture. Fluency is flexible, accurate, efficient retrieval — not a race and not optional. Automaticity, in the Panel's language, is fast, accurate, and effortless processing of content information, which frees working memory for more complex aspects of problem solving.[72] Computational proficiency depends on sufficient and appropriate practice to develop automatic recall of addition and related subtraction facts, and of multiplication and related division facts; fluency with standard algorithms; *and* understanding of the commutative, associative, and distributive properties.[73] Concepts and algorithms reinforce one another. Each also needs different experiences, including practice.

The 2021 how-to, in parent grain:

1. Identify already-learned topics. Facts the child can already get right untimed.
2. Keep the timed piece to 1–5 minutes. It is not the math hour.
3. Add it after many lessons on the idea, not on day one of making 10.
4. Support the learning: the ten-frame, the line, the factor family, a chance to fix a miss without shame.
5. Skip mere timed worksheets and unsupported computer drills as the programme.

A 2009 intervention guide asked for about ten minutes per intervention session on fluent retrieval. That is an intervention-session figure, not a parent-hour law.[74] The portable piece is a short, regular retrieval block, not a once-a-month speed test.

Making 10 is the 5–7 structure that makes facts knowable. 8 + 2, 7 + 3, 6 + 4, 5 + 5, and the cousins that make 10 another way. Doubles: 4 + 4, 6 + 6. Near-doubles: 6 + 7 as 6 + 6 + 1. Count-on from the larger: 8 + 2 as two hops from 8. Related

[72]National Mathematics Advisory Panel, *Foundations for Success* (2008), Chapter 5, automaticity definition, as in note 69.

[73]National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 11, as in note 69. Finding 10: conceptual understanding, computational fluency, and problem-solving skills belong together; debates ranking them are "misguided."

[74]Gersten et al. (2009), Recommendation 6, as in note 70.

subtraction: if 7 + 3 = 10, then 10 − 3 = 7. Write those as families, including `10 = 7 + 3`. Untimed, with objects, until they are right. Then a one-minute retrieval of the known pile.

A factor family is the 8–10 cousin, after equal groups are real. 3, 4, and 12: $3 \times 4 = 12$, $4 \times 3 = 12$, $12 \div 3 = 4$, $12 \div 4 = 3$. Arrays you can see. Then retrieval of the ones that are already right. Skip-counting by 2s, 5s, and 10s is a bridge, not a timed test and not a substitute for equal groups. Chapter 5 will own "is this equal groups?" This chapter will not pretend a skip-count song is multiplication fluency.

Multi-digit fluency is a different object. 27 + 15 as tens and ones, then a written method the child can explain, is Chapter 2 and Chapter 3 work becoming smooth. It still wants why-regrouping. Forty identical column problems with no talk is not Rec 6.

Retrieval can be oral, on paper with answers covered, or a homemade deck. A model may generate a sprint sheet for *you* the night before; you check a sample against a known table. The child never sees the key. Flash retrieval is not a language-model job during the attempt. Commercial apps exist. This book does not crown one. Oral, paper, or a dumb deck is enough.

Wrong answers on a timed minute are information: that fact was not yet known, or the clock arrived too early. Move it back to the untimed pile. A fast wrong answer is not "close." It is timing-too-early, or a mix-up you can diagnose (7 + 5 as 11, a count-on miss; 8×7 as 54, a neighbor fact). Untimed accuracy, then the clock on the known pile only.

Wrong answers you should be able to hear

1. *A fast 11 for 7 + 5, offered as fluency.* Fast and wrong. Timing-too-early, or the fact was never known. Delay the clock. Build 7 + 5. Untimed until it is right.
2. "*I'm not a math person; I'm slow.*" Speed as identity, talent story. Hear the feeling. Name effort: facts you know can become quicker with short practice. The timer is not a verdict on the person.
3. *A one-minute sheet of mixed facts the child cannot yet get right untimed, treated as the lesson.* Timing used to introduce. Rec 6's constraint was skipped. Sort the pile: known, almost, not yet. Time only known.

4. *Skip-count song offered as multiplication fluency, before equal groups are real.* A bridge wearing fluency clothes. Count by 2s on a walk is useful. 3 $\times$ 4 still needs three groups of four. Chapter 5.

5. *Forty identical problems, or a leaderboard, as the identity of the hour.* Volume and rank doing the work of retrieval. One to five minutes, known facts, then the rest of the hour is meaning, mixed practice, an exit ticket.

A sixth you will also hear: they are accurate untimed and frozen when the sand starts. The clock became a threat. Shorter. Three known facts. You hold the timer without staring. Tomorrow, the same three. Accuracy stays; the freeze usually thaws when the pile is truly known and the minute is truly small.

Five-minute parent warm-up

Do this before the lesson, on a scrap of paper, no child in the room.

Minute 1. Write three facts you are sure the child already gets right untimed. If you cannot name three, the clock is not this week's job. Making 10, or a double, is this week's job.

Minute 2. True or false, said aloud: timing creates the fact. Timing practices a known fact. Fast and wrong is fluency. Accurate first, then sometimes timed.

Minute 3. Imagine a one-minute sheet of twenty mixed facts, half unknown. Decide what a race would do. Decide what sorting into "known / not yet" would do instead.

Minute 4. Write 7 + 3, 10 − 3, 3 + 7. Same family. Write 10 = 7 + 3. The equal sign from last chapter still holds inside a fact family.

Minute 5. Write the sentence you will actually say: "Which of these do you already know for sure?" Put the pencil down. That sentence is the lesson.

If you can do those five minutes, you are ready to sit down. The child says the known ones. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version. Here is the shape scaled to this idea.

Warm-up (2–5 minutes). Known facts, oral or on a mini number line. Already-right material. Sometimes timed, once accurate — 1 minute, not 5, until the minute is easy. 5–7: making 10. 8–10: a factor family, if equal groups have begun.

Short model (3–7 minutes). One structure that makes facts knowable: a ten-frame for 7 + 3, or an array for 3×4 if you are in that band. Think-aloud. Write the family, including a total on the left. Stop. You are showing a known fact's meaning, not delivering a lecture titled *Fluency*.

Student attempt (8–15 minutes). Untimed accuracy on today's family. The child builds, writes, says. You wait. Struggle before rescue: ask, wait, hint ("make a ten"), then a short model on *your* frame, not as their fact.

One good question, then wait. "Which of these do you already know for sure?" A slow three. After they stop, wait again. Look at the cards, not at the child, if the silence is hard. Stahl's think-time; Rowe's windows were science class.[75]

Mixed practice (5–10 minutes). Yesterday's known facts mixed with today's new untimed family. A join story that uses a known fact. Mixing is how the child learns *when* to retrieve and *when* to build.

Exit ticket (2–4 minutes). Two to four items: two known facts, one almost, one from last week's equal-sign work. Done-enough is right, or wrong-with-a-reason we can use tomorrow. If you timed today, the ticket is untimed.

Exact wording you can say

"Which of these do you already know for sure?"

"Show me with objects or a line."

"Can you get it right if we don't time it?"

"If we time just these known ones for one minute, what happens?"

"What would happen if we timed ones you don't know yet?"

"You can change which ones are in the known pile."

When they are fast and wrong:

[75] Mary Budd Rowe, "Wait-Time and Rewards as Instructional Variables," NARST, April 1972, ERIC ED061103. Elementary science class. Not a mathematics RCT. Robert J. Stahl, "Using Think-Time and Wait-Time Skillfully in the Classroom," ERIC Digest ED370885, 1994. About three seconds as a convention; classroom teachers generally.

"I hear 11. Let's build 7 + 5. The clock can wait."

When they freeze at the timer:

"Same three facts. No clock. Then maybe a very short minute tomorrow."

When you are about to take over:

"Your voice. I'll wait."

Age-band moves: 5–7 / 8–10

5–7. Addition and related subtraction within 10, then 20. Making 10. Doubles. Count-on from the larger. Fact families that include `10 = 7 + 3`. Untimed with ten-frames and a line until accurate. Then 1 minute on a tiny known pile, oral or homemade cards. Skip-count by 2s and 10s on a walk as rhythm, not as a test. Speech is talk. You write the family if writing fights the sitting.

8–10. Keep addition/subtraction retrieval as mixed review. Multiplication and related division facts once arrays and equal groups are real (Chapter 5). Factor families. Timed retrieval of the facts that are already right on paper, 1–5 minutes, not the hour. Multi-digit add/sub becoming smooth, still with why-regrouping. Skip-count by 2s, 5s, 10s, then 3s, as a bridge. A store glance: "can you make this without counting every one?" on a small known combination.

A child of nine who is fast and wrong on 7 + 5 is still in untimed 5–7 work for that fact, whatever the birthday. A child of six who knows making-10 facts untimed is not "too young" for a one-minute known pile.

First try-it for the student

Three scraps on the table, facts you believe are known:

Say: "Which of these do you already know for sure?"

Wait. If they grab all of them, check untimed. If a miss appears, that one leaves the known pile. If they freeze, the question was about certainty, not speed. Let them sort: known, almost, not yet. Time nothing today unless the known pile is clean and you have already had many untimed sittings on these facts. If you do time, one minute, only the known pile, then stop.

Later the same week, the diagnostic item:

the same three, plus two they do not know yet, offered as a timed mix.

If they guess, or go fast-wrong, the mix trained timing-too-early. Sort again. Known pile only.

How to fade help. First sitting: you sort known/not yet together, untimed. Second: they sort, you wait, you hint ("show it on the frame"). Third: untimed known pile, they say the facts. Fourth: one minute on that pile, if accuracy held. The word *fluency*, if it arrives at all, arrives after the known pile.

When to stop talking. When you hear yourself answering 7 + 3. When the sit has become a lecture titled *Why Facts Matter*. When the timer has eaten the hour. One good question beats five. Stop while they still have a known fact left in them. The timed piece is 1–5 minutes. The rest of the hour is not the timer.

Practice that actually builds learning

Blocked, for a new move. The day you introduce making 10, only making 10, untimed, ten-frames visible. The day you introduce a one-minute retrieval, only the known pile. Mixing unknown facts into a first timed minute makes the child guess.

Mixed, for when to use it. Later the same week, known facts next to a join story next to an equal-sign true/false. Mixing is the practice of choosing: retrieve, build, or check both sides?

One incorrect example to diagnose. A fast wrong answer treated as "almost fluent." Or a timed sheet of unknowns treated as the lesson. Hear timing-too-early. The repair is a better question, then the ten-frame, then — if needed — a named fact after they build it. Silence in the face of a fast 11 is not kindness. Ranking the child with the timer is not Rec 6.

These are illustrations, not reported families.

Named try-it: Known-fact retrieval (in-home)

Time. One to five minutes, after untimed accuracy. Then stop. The rest of the hour is not this.

Materials. Oral, covering answers, or a homemade deck. Optional: a one-minute sand timer. A scrap for a simple self-tally if that helps.

Safety. Tone. Not a ranking. Not identity. Not a sibling contest. The youngest does not chew the cards. A parent who feels rusty and snaps at a miss can transmit tightness; let the child say the fact.[76]

The fun. Beating yesterday's self, if a simple self-graph helps. Being the person who knows 7 + 3 for sure.

The skill. Retrieve what is already right. Accurate first.

Say: "Which of these do you already know for sure?" Sort. One minute on that pile only. Count rights, not rank. A miss returns to untimed. Five-to-seven: making 10 and doubles. Eight-to-ten: a known factor family, or known add/sub mixed.

Named try-it: Making 10 / factor family (in-home)

Time. Five to eight minutes.

Materials. Ten-frame you draw, or an array of counters. Scrap. Number line.

Safety. Ordinary table.

The fun. The pattern. Seeing that 7 + 3 and 6 + 4 both make 10, or that 3×4 and 4×3 are the same rectangle.

The skill. 5–7 making 10. 8–10 a factor family, once equal groups are real. Untimed first. Write `10 = 7 + 3`. Write `12 = 3 × 4`. Kitchen table. The frame and the equation still happen.

Say: "Show me a way to make 10." Then another way. Then: "Which of these do you already know for sure?" The ones they can say without the frame go toward tomorrow's known pile. The ones they still build stay untimed.

[76] Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock, "Intergenerational Effects of Parents' Math Anxiety on Children's Math Achievement and Anxiety," *Psychological Science* 26, no. 9 (2015): 1480–1488. First- and second-graders of math-anxious parents learned less math over the year only when those parents reported frequent homework help. School homework-help study, not a homeschool RCT. Used here as a caution to let the child say the fact.

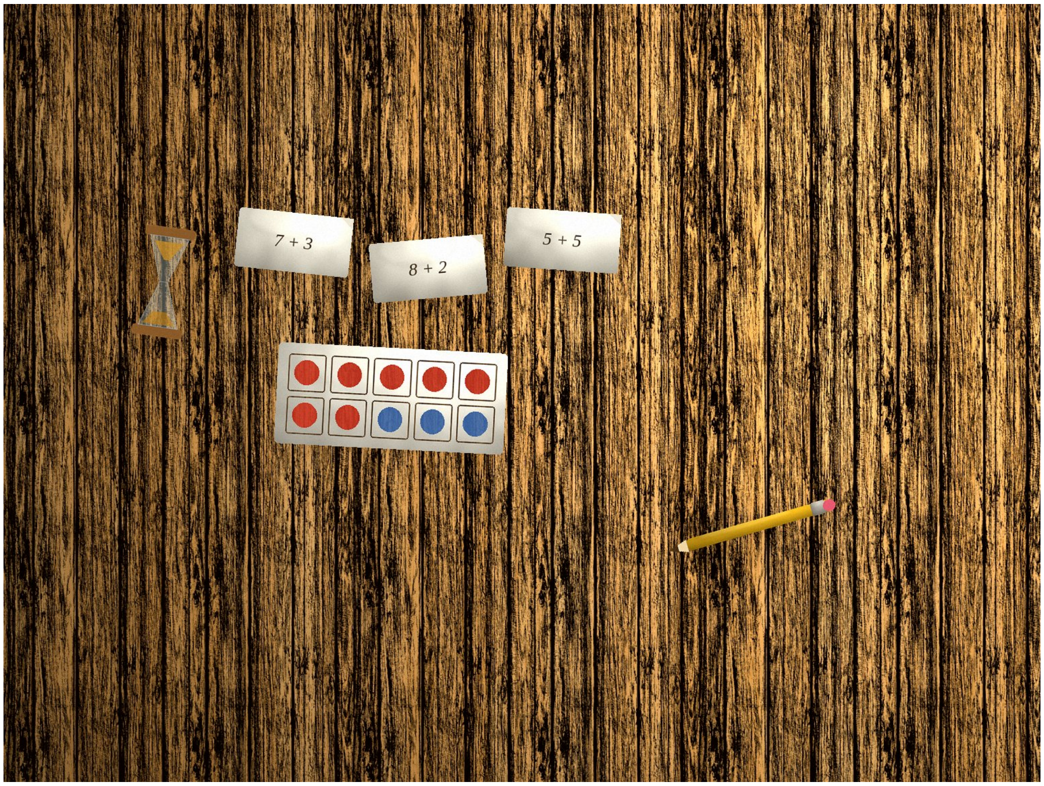

Figure 10: A one-minute sand timer next to three homemade scraps (7 + 3, 8 + 2, 5 + 5); a paper ten-frame filled as 7 and 3; no branded deck; no leaderboard; no people.

Named try-it: Walk skip-count (out-of-home)

Time. Three to eight minutes of a walk you were taking anyway.

Materials. Steps, fence posts, a rhythm.

Safety. Roads. Stay together. The child is counting steps, not crossing alone. No public quiz of a passerby.

The fun. The rhythm. Being the counter by twos, fives, or tens.

The skill. Skip-count as a bridge to equal groups, not as a timed test. Five-to-seven:

2s and 10s. Eight-to-ten: 5s, then 3s, still a bridge. Chapter 5 will ask whether this is equal groups.

Say: “Let’s count the posts by twos. Which of these jumps do you already know for sure?” If they freeze, slow the walk. If they race and skip a post, one-to-one from Chapter 1 is still in the room. A walk is not a trial of outdoor education. It is rhythm in a line you were already in. At home, write two of the facts the rhythm used.

Named try-it: Store, can you make this without counting every one? (out-of-home)

Time. Three to five minutes.

Materials. A small set already known: a pack of 6, two packs, a row of 5. A generic tag.

Safety. Aisle courtesy. A price is a number. It is not a sermon. No shoplifting a “test.” No public quiz of a cashier. No race in public.

The fun. The glance. Seeing 5 + 5 as 10 on two packs, if fives are known.

The skill. Known combinations, not a race in public. Five-to-seven: small whole numbers they already subitize or make-10. Eight-to-ten: a known double or a known 5-fact. If they do not know it for sure, count. Counting is legal. Guessing in the aisle is not fluency.

Say: “Which of these do you already know for sure? Can you make this without counting every one?” If they guess, count together quietly. The aisle is motivation. The known pile still gets written at the table.

Blocked / mixed, in this week. Monday: making 10, untimed only. Tuesday: sort known/almost/not yet. Wednesday: one-minute known pile, if accuracy held. Thursday: walk skip-count. Friday: mix known facts with an equal-sign true/false and one diagnostic fast-wrong.

Talk box

This is an illustration, not a reported family. Three scraps sit on the table. The parent has not announced a race.

Opening question

"Which of these do you already know for sure?"

Not: "How fast are you?"

Follow-ups (pick three to five; you will not use all of them every sitting)

1. Show me with objects or a line. How do you know this one?
2. Can you get it right if we don't time it? Show me with things, a sketch, the number line, a number sentence.
3. If we time just these known ones for one minute, what happens? What would happen if we timed ones you don't know yet?
4. Does that answer make sense? You said 11 for 7 + 5; let's build it.
5. You may change your mind. You can change which ones are in the known pile. Try it another way.

How to wait

Ask. Count a slow three in your head. Look at the scraps, the ten-frame, or the timer, not at the child's face, if the silence is hard. After they stop talking, wait again. If they are mid-reason, do not cut them off. Three seconds is a convention, not a sacrament — Stahl's think-time. Rowe's windows were science class. Use the pause. Neither is a homeschool trial of fluency.

What a stuck silence usually means

They think speed is the identity, and they can feel that certainty was asked for. Or they guess. Or timing was day one. Or they are accurate and frozen. Or wait-time after the question was zero. Or they are hunting for the "math person" answer. Next move: smaller known pile, untimed, wait, point back at the frame. "Show this one?" is a hint. "It's 10" said by you is not a hint; it is the grab. Done enough this week: a spoken sentence. "I know 7 + 3 for sure. I still build 7 + 5."

For the student

This page is for you.

Which of these do you already know for sure?

That is a question about facts you can get right without a clock. 7 + 3, if you know it. 5 + 5, if you know it. A fact you still have to build is a fact you are still learning. That is allowed. The clock waits.

Fluency means you can get a known fact without rebuilding it every time. It does not mean you are a fast person. It does not mean you are a slow person. It means this fact is known, and that one is not yet.

A tiny worked example

Three scraps: 5 + 5, 8 + 2, 7 + 5.

You know 5 + 5. You know 8 + 2. You still build 7 + 5 on a ten-frame.

The known pile is 5 + 5 and 8 + 2.

If you time anything, you time those two, for one minute, after you already get them right with no clock.

If you say 11 for 7 + 5, fast, that is not fluency. Build 7 + 5. Put it in the "not yet" pile. You may change which pile a fact lives in. That is not losing.

Two tries

1. Sort three real facts into known, almost, not yet. Show a known one with a ten-frame or an array anyway, once. Then say it without the frame.
2. A walk by twos, or a small set at a store you already know. Which jumps or combinations do you know for sure? If you do not know, count. Counting is allowed.

Write nothing if writing fights the sitting. Talk the known ones. If you are older and writing is easy, you may write: "I know ___ for sure. I still build ___."

Explain it back

Tell someone at the table, in your own words, which facts you know for sure, and how you know you know them. Then say what a timer is for, and what it is not for.

Challenge

Someone times facts they do not know yet, and calls the guesses fluency. What did they use? How do you sort known from not yet? Someone says "I'm slow, so

I'm not a math person." What did they use? How do you practice a known fact for one short minute without making the timer the whole of you?

You are allowed to struggle. You may use a ten-frame, a line, homemade cards, a walk. You say the known ones. You build the rest. If you get stuck, ask for a hint — not the finished number. Then try again.

When you talk, a sentence about the known pile is enough for today. You do not have to write a paragraph. You do not have to call yourself fast. Today you pick three facts you know for sure and say them.

A picture a computer made of a stopwatch is a scene. It is interesting. It is not your known pile.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The child is fast and wrong, or the clock arrived before the facts were known.

Timing-too-early is doing the work of fluency. Next move: sort the pile. Untimed accuracy on a tiny set. Ten-frame or array in sight. Delay the clock until those facts are right without it. Stay here if this is still the bottleneck after several weeks of short sits. Equal groups in the next chapter will not hold on top of guessed products, and two-digit work will not hold on top of guessed 7 + 5. A human who will sit with a ten-frame and wait, not an app that ranks, is a reasonable next step if you have tried the known-pile work and fast-wrong is still the whole hour.

2. The child is accurate untimed and frozen, or treats speed as identity.

"I'm not a math person." Or they know 7 + 3 and cannot say it when the sand runs. The timer became a threat, or talent-talk did the teaching. Next move: no clock for a stretch. Three known facts, oral, you looking at the cards not at them. A simple self-tally only if it helps. Name effort: this fact became quicker because we practiced the ones you already knew. Slow down the race talk until a known fact can be said in a quiet minute. A child who can build every fact and cannot yet retrieve any is not ready to skip untimed accuracy — and they *are* ready to start a tiny known pile, even one fact. A tutor is useful if freeze or talent-talk remains the default after a couple of weeks of daily short known-pile work *and* the hour has become a fight.

3. The child will not say a fact, or every sitting ends in a shrug, and the silence after your question is a wall.

The wait was zero, or the question was “how fast are you” in disguise, or the pile is all unknowns, or they are guessing what you want. Next move: shorter sits, one known fact, the opening question from the talk box, wait a slow three twice. If silence is hard, you look at the scrap, not at them. If you hear yourself answering 7 + 3, you have started doing the work. If they offered a guess, name the miss and build it. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that fills the silence, is the release valve.

When to slow down. Fast-wrong still wearing fluency clothes. Timer still the identity of the hour. Unknowns still in the timed pile. Sits so long that building never starts. Those are brakes. “Not ready” because of age is the brake this book will not use. “Not ready” because the fact is not yet known untimed is a real brake. A fifty-item sheet is not a reason to skip three known facts.

When to go ahead. The child can sort known from not yet. Three facts are right untimed. A short minute on that pile has happened without becoming the hour, or you honestly delayed the clock because freeze was in the room — both can be done-enough. Making 10 or a factor family has been built, then said. A walk or a store glance used a known combination without a public race. Short sits are ordinary. Then equal groups have somewhere to sit. Being “good at facts” because a child is willing to race is not a reason to skip accuracy.

When to get a human tutor. You have run the known pile, or the ten-frame, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear a fast wrong answer as timing-too-early. Outsourcing the hearing is the thing to avoid, not asking for help.

Correct a fast 11 without crushing the attempt. “I hear 11. Let’s build 7 + 5. The clock can wait.” Hearing a wrong answer, asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The child says the known facts first. You hold the pile and the timer. A tool may explain today's idea *to you* from this chapter, generate a sprint sheet the night before with the key on *your* page, write a short script for *you* to say after the child has tried, offer a hint after an attempt, or help you diagnose a miss the child already produced. You check a sample of any generated sheet against a known table. The child never sees the key. Crop to the paper. Do not upload the child's face.

Ages 5–7: no live open chat as the child's partner. Ages 8–10: still parent in the room. Unaided first. SCRIPT after the try.

Leave these out of the hour: a chatbot as the only partner; a tool that answers 7 + 3 during the attempt; a camera pointed at a worksheet so a solution pops up; a detector score; a certificate of fluency; a tool asked to invent which facts this child knows. Look at last week's exit tickets.

Flash retrieval is oral, paper, or a homemade deck. A language model will happily print fifty mixed facts. Treat every model-supplied "sprint" as untrusted until you have sorted known from not yet. In a high-school math study, an unguarded chatbot made practice look better and left students worse when the window was closed. That paper is high-school mathematics, not a 5–10 trial.[77] If the tool produces the fact, the child is a spectator.

This book does not crown a commercial flashcard app. Retrieval can be a sand timer and three scraps.

A ten-frame, a number line, homemade cards, and a one-minute timer are tools too. Use them, then fade the frame when the fact is known. Bring the frame back when a miss appears.

[77] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. GPT Base +48% on assisted practice, −17% on the unaided exam, high-school mathematics, Turkey, nearly 1,000 students, grades 9–11. Not a 5–10 trial. The sitting condition — a tool that produces the response — is the mechanism used here.

What “done enough” looks like

Placement is by skill, not birthday. A “grade 2 math workbook” is a publisher’s scope, not a legal grade, and not a transcript line. The record, when you need one, is a dated notebook, the task named, plus an exit ticket, plus one diagnostic item. The title a stranger can read is Mathematics, or Arithmetic. It is not Little Thinkers Math I, and it is not a fluency-certificate brand.

Checklist before moving on

- You can hear a fast wrong answer as timing-too-early, and you can delay the clock.
- The child can sort a small pile into known, almost, and not yet.
- Three facts are right untimed, shown at least once with a ten-frame, an array, or a line.
- Timed work, if it happened, lasted 1–5 minutes, only on known facts, and was not the lesson.
- Making 10 (5–7) or a factor family (8–10, once groups are real) has been built and then said.
- Related facts were written as a family, including a total on the left (`10 = 7 + 3`).
- At least one out-of-home rhythm or glance has happened: skip-count on a walk, or a known combination at a store, without a public race.
- Speed was not the identity of being good at math. A simple self-tally, if used, beat yesterday’s self, not a sibling.
- Sits can be short. A spoken known pile is enough. You did not require a fifty-item sheet as the proof.
- You can hear fast-wrong, freeze, talent-talk, and a skip-count offered as multiplication, and you can ask a good question, without answering 7 + 3.

If most of that list is true, go on to equal groups, even if the birthday says otherwise. If the birthday says “third grade” and 7 + 5 is still a guess against a clock, stay. The next chapter asks whether this is equal groups, or something else. It needs facts that are known, and groups that are real, not a times-tables programme as day one.

A path through elementary mathematics is the promise. A diploma is not. A percentile is not. A timer is not.

Chapter 5

Equal groups: multiplication and division

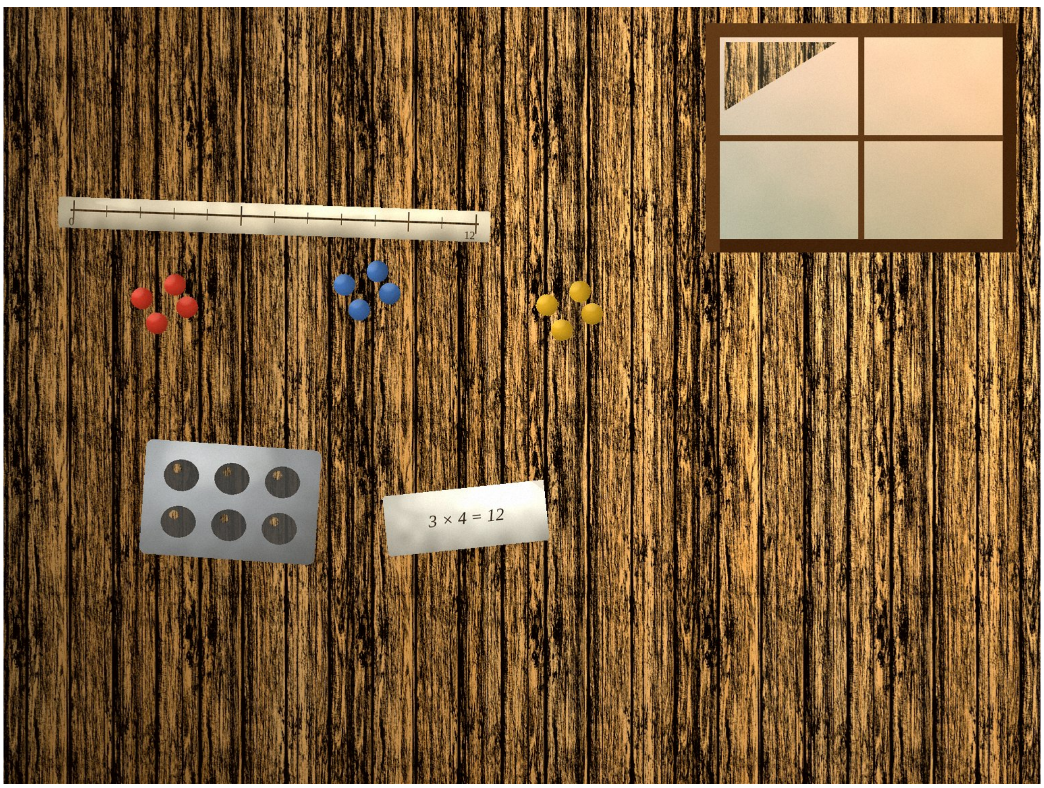

Figure 11: A kitchen-table still-life from a high three-quarter view: three small circles of four counters each, a muffin tin with equal wells, a scrap of paper with the handwritten equation $3 \times 4 = 12$, and a short number line. No people. No logos. No keyword poster.

Why this matters

Is this equal groups, or something else?

That is not a vocabulary quiz. It is not a cue to recite a times table. It is the question that turns “four cartons of twelve eggs altogether” from a word hunt into a picture you can draw. Three groups of four crackers is equal groups. Eight crackers and then three more is a join. Twelve crackers, four already eaten, is a separate. Two piles, and how many more in one than the other, is a compare. The cover story can hide the structure. The structure is the mathematics.

This chapter is multiplication and division as equal groups, not as a times-tables programme on day one. The product-unknown story asks how many in all: three bags of four, how many crackers. Measurement division — sometimes called quotitive — asks how many groups: twelve crackers, four in each bag, how many bags. Partitive division asks the size of a group: twelve crackers into three bags, how many in each. Those three stories are cousins. They share the same sketch. They write matching number sentences. Arrays and area come later in the eight-to-ten band, once the groups are real. Skip-count and share live in the five-to-seven band. Fact fluency, once the groups are real, lives with Chapter 4. The tables are not the door.

What this idea unlocks is almost everything later that looks like “times” or “share.” Fractions as fair shares sit on the same picture. Area as covering sits on an array. Two-step stories in the next years sit on a type you can name. If “altogether” still means add, four cartons of twelve become sixteen, and the child is not being silly. They used a word the way a keyword list taught them to use it. The Institute of Education Sciences intervention guide for struggling elementary students, Recommendation 5, is the household picture of that miss: identifying a problem type is different from grabbing an operation, and it is not useful to glue a type to an operation. Their Table 5.2 is the trap you will actually hear. *Altogether* fails on equal groups. *Each* fails on partitive division. *Share* and *double* fail the same way.[78]

[78]Lynn S. Fuchs, Nicole Bucka, Benjamin S. Clarke, Barbara Dougherty, Nancy C. Jordan, Karen S. Karp, and John Woodward, *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*, WWC 2021006 (Washington, DC: Institute of Education Sciences, 2021), Recommendation 5 (strong evidence, 18 studies): teach students to identify word-problem types that include the same type of action or event; identifying a type is different from determining the operation; it is not useful to associate a type with an operation. Named types include Change, Combine, Compare, and Equal Groups. Table 5.2: *altogether* fails on “Alice bought 4 cartons of eggs with 12 eggs in each carton. How many eggs does Alice have altogether?” (4 + 12 = 16); *each*

Hear 4 + 12 for four cartons of twelve. Then draw four circles of twelve. Then write 4×12. That hearing is worth the struggle.

A second reason sits next to the first, labelled so it does not become a kitchen promise. Robert Siegler and colleagues found that elementary whole-number *division*, alongside fractions, uniquely predicted high-school algebra years later, after other elementary skills were controlled. Addition, subtraction, and multiplication were weaker or not significant once those controls were in. The finding is a prediction from two large surveys, not a classroom experiment proving a teaching method, and not a reason to rush invert-and-multiply onto an eight-year-old.[79] Division as "how many groups" and "how many in each" is the object here. Invert-and-multiply is not a first move in this book.

The National Mathematics Advisory Panel's Finding 10 still holds: conceptual understanding, computational fluency, and problem solving belong together.[80] A programme that only chants tables is selling a fragment. A programme that never writes $3 \times 4 = 12$ is selling a different fragment. The Common Core map puts equal-sized groups, arrays, and area models in Grade 3, and "know from memory all products of two one-digit numbers" by the end of that map. Those are coordinates, not a birthday and not a homeschool statute.[81] Fluency with multiplication

fails on partitive division ($75 \times 3 = 225$ for three trays, 75 blocks altogether, how many on each); *left*, *more*, *share*, and *double* fail in the same table. Scope: grades K–6, small-group or one-on-one intervention for students struggling, not a whole-class parent-hour randomised trial. Access date for URLs in these notes: 3 September 2026.

[79] Robert S. Siegler, Greg J. Duncan, Pamela E. Davis-Kean, Kathryn Duckworth, Amy Claessens, Mimi Engel, Maria Ines Susperreguy, and Meichu Chen, "Early Predictors of High School Mathematics Achievement," *Psychological Science* 23, no. 7 (2012): 691–697. UK Birth Cohort Study N = 3,677: a 1-SD increase in age-10 fractions uniquely associated with +0.15 SD high-school algebra and +0.16 SD total math after controls; US PSID-CDS N = 599: +0.17 SD algebra and +0.18 SD total math. Whole-number division was the other unique elementary predictor. Addition, subtraction, and multiplication weaker or non-significant after controls. Fractions and division generally not uniquely predictive of high-school literacy. Prediction finding, not a classroom experiment proving a teaching method, and not a kitchen promise.

[80] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Finding 10: conceptual understanding, computational fluency, and problem-solving skills are mutually supportive; debates ranking them are misguided. Finding 11: automatic recall of multiplication and related division facts, with understanding of the properties, frees working memory. Finding 15: "too young" / Piaget-stage claims wrong if prerequisites are present.

[81] National Governors Association Center for Best Practices and Council of Chief State School Officers, *Common Core State Standards for Mathematics* (2010), Grade 3 critical area: equal-sized groups, arrays, area models; multiplication as unknown product; division as unknown factor

and division of whole numbers by the end of Grade 5 is the *end* of this book's band, not Tuesday of week one.

You do not need to be a mathematician. You do need to hear 4 + 12 for "four cartons of twelve altogether" as a keyword grab, not a cute slip, and to ask whether this is equal groups or something else.

This week you can learn the three equal-groups stories — total unknown, number of groups unknown, size of a group unknown — well enough to hear the miss. Today the student can draw three groups of four and write $3 \times 4 = 12$.

For the parent: understand it yourself

Many adults feel rusty on why 3×4 and $12 \div 3$ are the same picture, said two ways. That is ordinary. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The child still draws.

Everyday picture. Three small bags. Four crackers in each. You can see twelve without counting every cracker from one, if the groups are equal and you trust the four. You can also start with twelve crackers and pack four to a bag until they are gone — three bags. You can start with twelve and three bags and share them out, one to each bag, around and around, until each bag has four. Same crackers. Three stories. The kitchen is the motivation. The number sentence is still the work: $3 \times 4 = 12$, and $12 \div 3 = 4$, and $12 \div 4 = 3$. They are matching stories, not three different subjects.

Precise picture. Multiplication, in this chapter, is equal groups with the product unknown. Three groups of four. How many in all. Write $3 \times 4 = 12$. Write $12 = 3 \times 4$ as well. The equal sign still means "the same as," as it did in Chapter 3. The total is the same amount as three groups of four.

Division here is the unknown-factor cousin of that same picture. If $3 \times \square = 12$, the box can be the size of a group or the number of groups, depending on the story.

Measurement division (quotitive): you know the size of each group; you want how

(unknown number of groups or unknown group size). 3.OA: by the end of Grade 3, know from memory all products of two one-digit numbers. Cited as a map of typical U.S. public-school placement, not a homeschool statute. NMAP Table 2: fluency with multiplication and division of whole numbers by the end of Grade 5 — the end of this book's band. The Panel noted there is no empirical research on the placement of those benchmarks; they should be interpreted flexibly.

many groups. Twelve crackers. Four in each bag. How many bags? You measure the twelve by fours. You can skip-count 4, 8, 12, and see three counts. You can make bags until the crackers are gone.

Partitive division: you know how many groups; you want the size of each. Twelve crackers. Three bags. How many in each bag? You partition the twelve into three equal shares. You can deal them out, one to each bag.

Those two divisions feel different in the hands. They write related sentences. 12 $\div$ 4 = 3 is "how many groups of four." 12 $\div$ 3 = 4 is "how many in each of three groups." A child who can only chant "twelve divided by three is four" has a fact. A child who can say which story they are in has the mathematics.

An *array* is equal groups stood in rows. Three rows of four is still 3×4. Four columns of three is still 12, and commutativity starts to have a picture: 3×4 is the same amount as 4×3 because the array did not change when you turned your head. Arrays belong later in the eight-to-ten band, once groups of counters are easy. *Area* as covering a rectangle with unit squares is a further step on the same picture, still in that later band, not a five-year-old first sitting. Skip-counting by twos and fives is a bridge in the five-to-seven band. It is not yet the type.

A strip diagram — a bar split into equal parts — is a picture the Institute of Education Sciences problem-solving guide, written for grades 4 through 8, rates as a strong recommendation for visual representations.[82] That guide is on-age for the eight-to-ten band. For five-to-seven, the honest picture is still counters in groups, then a sketch of those groups, then the equation. Bar models are a representation. They are not a franchise, and they are not a reason to skip the counters.

Wrong answers you should be able to hear

1. *4 + 12 = 16 on "Alice bought 4 cartons of eggs with 12 eggs in each carton. How many eggs does Alice have altogether?"* The word *altogether* was treated as a cue to add. Table 5.2 of the 2021 guide is this item. Identifying the type — equal groups, product unknown — is different from grabbing

[82] John Woodward, Sybilla Beckmann, Mark Driscoll, Megan Franke, Patricia Herzig, Asha Jitendra, Kenneth R. Koedinger, and Philip Ogbuehi, *Improving Mathematical Problem Solving in Grades 4 Through 8*, NCEE 2012-4055 (IES, 2012; revised 2018), Recommendation 3 (strong evidence): visual representations, including strip diagrams. Floor is grade 4 (~age 9–10). On-age for the 8–10 band. Recommendation 2 (monitor and reflect) moderate; Recommendation 1 (prepare problems) minimal. Strip diagrams are the honest home for bar models as a representation, not a curriculum trial.

addition because a familiar word showed up. Draw four circles. Put a 12 in each, or twelve marks. Write 4×12. The keyword is named as a trap, once. It is not replaced with a different poster of cue words.

2. *$75 \times 3 = 225$ on "Miles had 3 trays with the same number of blocks on each. 75 blocks altogether. How many on each tray?"* The word *each* was treated as a cue to multiply. This is partitive division: 75 into 3 equal trays. The type is equal groups with the group size unknown. $75 \div 3$, or $3 \times \square = 75$. The child who multiplies is not being lazy. They glued a word to an operation.

3. *The child skip-counts 2, 4, 6, 8, 10 fluently and cannot show two groups of five, or five groups of two.* Skip-count is a rhythm. It is a useful bridge. It is not yet equal groups. Ask them to make the groups with crackers. If the groups will not form, stay with sharing and packing. The chant can wait.

4. *"Division is the one with the little house."* The symbol was learned as a drawing, not as an unknown factor. The house can wait. Write $12 \div 3 = 4$ and $3 \times 4 = 12$ on the same page. Point at the groups. The symbol is a record of the story, not a piece of furniture.

5. *Leftover crackers treated as "subtract the friends" or as a reason the problem is broken.* Remainder is what is left when equal groups do not use everything. Two leftover is two leftover. It is not a social story. Name it. Write it. In the five-to-seven band, leftover named without shame is enough. In the eight-to-ten band, $14 \div 4 = 3$ remainder 2, or 3 groups of 4 with 2 left, is the same picture with a number sentence.

A sixth you will also hear: a recitation of the 4-times table offered as if the child can tell a product-unknown story from a measurement story. Facts are welcome later, as Chapter 4 said, once they are already right untimed. They do not replace the sketch. The National Panel wanted automatic recall *and* the meaning.[83] Sequence: groups, sketch, equation, later the known facts, sometimes timed.

Five-minute parent warm-up

Do this before the lesson, with twelve crackers or twelve scraps of paper, no child in the room.

Minute 1. Make three groups of four. Say out loud: "Three groups of four. How

[83] NMAP, *Foundations for Success* (2008), Findings 10 and 11.

many in all?” Write $3 \times 4 = 12$. Write $12 = 3 \times 4$.

Minute 2. Push them back into one pile. Pack four to a bag. Count the bags. That is measurement division. Write $12 \div 4 = 3$. Say: “How many groups of four.”

Minute 3. One pile again. Three plates. Deal them out. That is partitive division. Write $12 \div 3 = 4$. Say: “How many in each of three groups.”

Minute 4. Look at “4 cartons of 12 altogether.” Decide, without filling an answer, what 16 would mean. You are installing the miss so you can hear it tomorrow. The right picture is four groups of twelve.

Minute 5. Write the sentence you will actually say: “Is this equal groups, or something else?” Put the pencil down. That sentence is the lesson.

If you can do those five minutes, you are ready to sit down. The child makes the groups. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version, including the talk-box shape, the in-home and out-of-home try-it shape, and the AI rules. Here is the shape scaled to equal groups.

Warm-up (2–5 minutes, unaided). Two facts the child already knows from Chapter 4, if any are known. One small equal-groups picture already made yesterday: “Two groups of three. How many?” No device. No table to chant.

Short model (3–7 minutes). One story, one picture, one written sentence. Three bags of four, built, sketched, then $3 \times 4 = 12$ said as “three groups of four is the same as twelve.” You talk for a few minutes. Then you stop.

Student attempt (8–15 minutes). One to four items of *today’s type*. Product unknown on Monday. Measurement on Wednesday. Partitive on Thursday. The child builds, sketches, writes. You wait. Struggle before rescue: ask, wait, hint, then model a *different* item. Leave the pencil in the child’s hand.

One good question, then wait. “Is this equal groups, or something else?” Not “what’s the answer?” Not “what does altogether mean?” After you ask, a slow three is a convention, not a sacrament. Robert Stahl called that silence think-time and wrote for classroom teachers. Mary Budd Rowe measured pauses in elementary *science* class. Use the pause. Neither is a homeschool math trial.[84]

Mixed practice (5–10 minutes). Yesterday's join or separate next to today's groups.[85] A compare next to a product-unknown. The 2021 guide's first recommendation includes mixing previously and newly learned material. That is the warrant for mixing inside the hour, not a reprint of a grade-7 interleaving trial.[86] If the book on the table is almost entirely blocked — most are, the week a new operation arrives — you pull one older item onto the page.

Exit ticket (2–4 minutes, unaided). Two items. One is today's type. One is yesterday's. Done-enough is "right, or wrong-with-a-reason we can use tomorrow," not a percentage.

That shape is a practice you impose on whatever crackers, muffin tin, or store tag is already in the week. It is not a 180-day times-tables calendar.

Exact wording you can say

On type:

"Is this equal groups, or something else?"

"What is in each group? How many groups?"

"Are we looking for the total, the number of groups, or the size of a group?"

On the picture:

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[84] Robert J. Stahl, "Using 'Think-Time' and 'Wait-Time' Skillfully in the Classroom," ERIC Digest ED370885 (1994): think-time, eight categories of silence, ~3 seconds as a convention, written for classroom teachers generally. Mary Budd Rowe, "Wait-Time and Rewards as Instructional Variables," NARST paper, 1972, ERIC ED061103: elementary science class; typical wait ~1 second after a question and ~0.9 seconds after a student answer; extending both to 3–5 seconds changes discourse. Use the pause for the math hour. Neither is a homeschool math randomised trial. Do not mash Rowe and Stahl as a math trial.

[85] Russell Gersten et al., *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools*, NCEE 2009-4060 (IES, 2009), Recommendation 4 (strong): instruction on word problems based on common underlying structures. Recommendation 6 (moderate): about 10 minutes per intervention session on fluent retrieval — an intervention-session figure, not a parent-hour law.

[86] Fuchs et al., *Assisting Students Struggling with Mathematics* (2021), Recommendation 1 (strong evidence, 43 studies): systematic instruction, including mixing previously and newly learned material. The warrant for mixed practice inside the math hour. Doug Rohrer's interleaving trials are grade 7; not cited here as a 5–10 finding.

"Show me with the crackers. Then a sketch. Then write it."

"Three groups of four. Write $3 \times 4 = 12$. Write $12 = 3 \times 4$ too."

On a keyword that showed up:

"Altogether showed up. Does that mean add *this* time?"

"Each showed up. Are we making more, or sharing out?"

On the two divisions:

"We know how many in each bag. We want how many bags. That is measuring by groups."

"We know how many bags. We want how many in each. That is sharing out."

On leftover:

"Two are left. That is leftover. It is still two. We can write it."

When you are about to take over:

"Your crackers. I'll wait."

Then wait. Look at the groups, not at the child's face, if the silence is hard. After they stop, wait again. If they are mid-count, do not cut them off. A follow-up that arrives in half a second trains them to wait *you* out.

Age-band moves: 5–7 / 8–10

5–7. Speech is talk, not an essay. The groups are small: twos, fives, tens, then threes and fours. Skip-count on a walk is a bridge, not a test. Share a snack and name leftover. Pack a small pile into bags of two or five. Write $2 \times 5 = 10$ and $10 = 2 \times 5$. Product unknown is the first story. Measurement ("how many bags of two in this pile") can arrive the same week as packing. Partitive ("share among two plates") is the snack they already do. Arrays can wait. Area can wait. The little division house can wait. You write the caption if writing fights the sitting. The equal sign still means the same as.

8–10. Same three stories, larger numbers, then two steps. Arrays: three rows of four, then turn the paper. Area as covering, once the array is easy. A strip for a compare that sits next to an equal-groups story in mixed practice. Remainder written. Matching sentences: $3 \times 4 = 12$, $12 \div 3 = 4$, $12 \div 4 = 3$, on one page, with a sketch. Fact families after the groups are real — Chapter 4's timed minute only on facts already right untimed.[87] A two-step store story: three apples at forty cents, then how much change from two dollars, still named by type, not by a keyword. Strip diagrams are fair in this band. Counters do not have to leave the table the day the strip arrives.

Not a stable skill at either band: invert-and-multiply; cross-multiplication; a 180-day table-chant as the identity of being good at math. Same faculty. Different grain. Not two species of child.

First try-it for the student

Twelve crackers or twelve scraps, no vocabulary card:

three groups of four, already in reach.

Say: "Is this equal groups, or something else? How many in all?"

Wait. If the child counts every cracker from one, they have not yet used the groups. Hear the count. Ask whether the groups are equal. Ask whether they can skip-count the groups. If they write $3 + 4$, they joined the numbers in the sentence instead of the groups. Build it again. If they write $3 \times 4 = 12$, they heard the type. If they stare, model on a *different* small set — two groups of three, said as "two groups of three is the same as six" — then return to three groups of four.

Later the same week, the diagnostic item:

"4 cartons of 12 eggs altogether. How many eggs?"

Hear 16. Hear 48. Hear a freeze. After the child has attempted, you may show a worked *incorrect* example that wrote $4 + 12 = 16$ and ask what that person thought *altogether* meant. Generation first, then the named error. The child names the trap. You do not narrate it as a verdict on talent.

How to fade help. First sitting: you build together, you say the type, they echo the sentence. Second: they build, you wait, you hint ("what is in each group?"). Third: they build, sketch, and write. Fourth: a cousin of the same type — measurement instead of product unknown — with the first sketch closed. The times-table chant, if it arrives at all, arrives after the groups.

[87] Fuchs et al. (2021), Recommendation 6 (strong evidence, 27 studies): timed activities 1 to 5 minutes, on already-learned material, not the entire focus, not used to introduce concepts, not mere timed worksheets. Multiplication fact fluency sits here after the groups are real, as Chapter 4 already taught.

When to stop talking. When you hear yourself answering. When the sit has become a lecture titled *What Multiplication Is*. When you have already asked a second and a third question before they finished the first. One good question beats five. Stop while they still have a group left to make.

Practice that actually builds learning

Blocked, for a new move. The day you introduce product unknown, give a short set that is only that: three groups of four, two groups of five, four groups of two. The day you introduce measurement, only packing. The day you introduce partitive, only sharing out. Mixing too early makes the child hunt for a word instead of building the groups.

Mixed, for when to use it. Later the same week, a join from Chapter 3 reappears next to today's groups, and a compare next to a share. Mixing is the practice of choosing: is this equal groups, or something else? You impose that mix on the week you already have. Kitchen, walk, and store *are* the practice. They do not replace the sketch or the written equation.[88]

One incorrect example to diagnose. $4 + 12 = 16$ on four cartons of twelve altogether. Hear the keyword. Hear the add. The repair is a better question, then four circles of twelve, then — if needed — the named type and the sentence 4×12. Silence in the face of a keyword error is not kindness.

These are illustrations, not reported families.

Named try-it: Snack packs (in-home)

Time. Eight to twelve minutes. Then stop.

Materials. Three bags of four crackers, or a muffin tin, or twelve pasta pieces and three cups. A scrap of paper.

[88]Douglas Frye, Arthur J. Baroody, Margaret Burchinal, Sharon M. Carver, Nancy C. Jordan, and Jonathan McDowell, *Teaching Math to Young Children*, NCEE 2014-4005 (IES, November 2013). Ages 3–6. Recommendation 4 (minimal evidence; panel still recommends): talk about math in everyday situations and link informal knowledge to symbols. Kitchen and store motivate; they do not replace the written equation.

Safety. Ordinary food hygiene. Choking hazard of small pasta and small crackers for the youngest. No shaming who ate what. Let the child move the objects. A parent who feels rusty and takes the pasta can turn the sitting into a performance; leave the pieces in the child's hands. That caution comes from a school homework-help study of first- and second-graders, not a homeschool trial. The useful move is still: they build.[89]

The fun. Making packs. Finding how many in all, how many bags, how many in each.

The skill. Three equal-groups types. Vary the unknown. Write 3×4 and $12 \div 3$ as matching stories, not as keywords.

Start with the bags already packed. "Is this equal groups, or something else? How many in all?" Then empty them. "Four in each bag. How many bags for these twelve?" Then three plates. "Share these twelve onto three plates. How many on each?" Same crackers. Three stories. Same week, not three years.

5–7. Twos and fives first if fours fight. Talk the sentence. You may write it.

8–10. They write $3 \times 4 = 12$, $12 \div 3 = 4$, $12 \div 4 = 3$. A leftover sitting: fourteen crackers, four to a bag. Three bags, two left. Name leftover.

[89]Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock, "Intergenerational Effects of Parents' Math Anxiety on Children's Math Achievement and Anxiety," *Psychological Science* 26 (2015): 1480–1488. First- and second-graders: parent math anxiety mixed with frequent homework help predicted worse child math at year's end; when help was infrequent, the association disappeared; math-specific, not reading. School homework-help study, not a homeschool randomised trial. The portable caution: leave the objects in the child's hands.

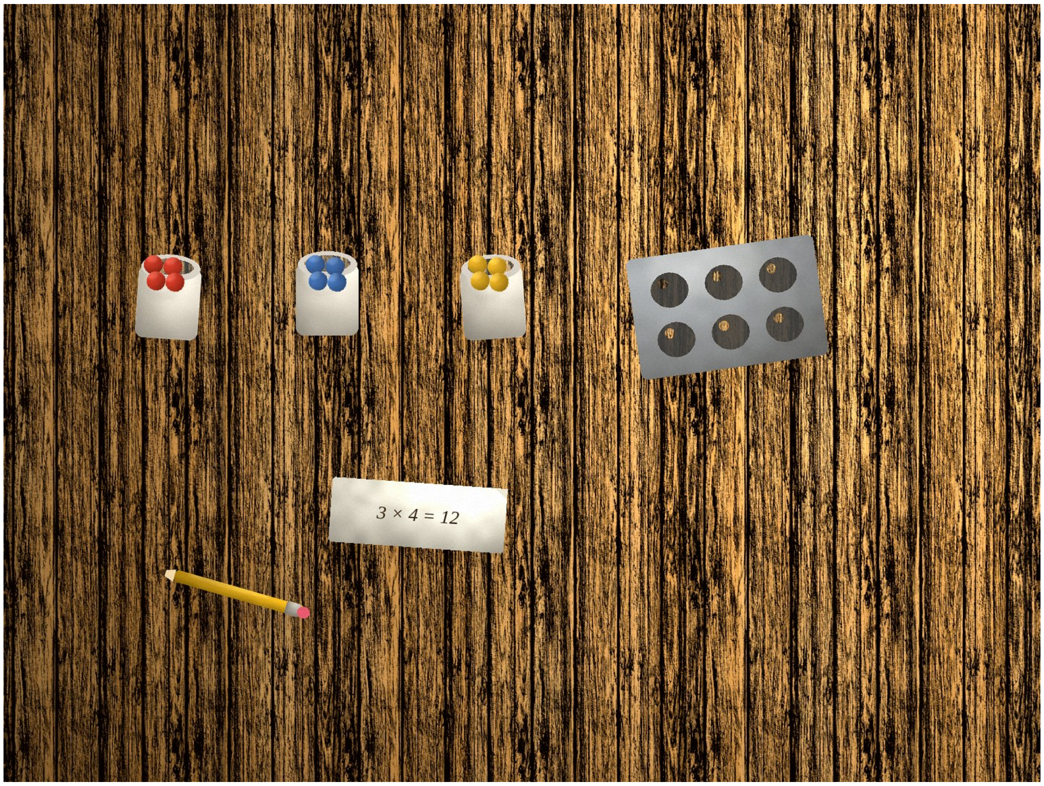

Figure 12: Three small bags or cups with four counters in each, a muffin tin beside them, and the equation $3 \times 4 = 12$ on a scrap. No people. No logos.

Named try-it: Recipe double (in-home)

Time. Ten to fifteen minutes, when you were already making something.

Materials. A half-cup already in the drawer. A recipe the family was making. A scrap for the equation.

Safety. No flame for a five-year-old. Ordinary cooking if the family already cooks. A kitchen fair share uses a measuring cup — not a nutrition sermon. The half-cup is a quantity. It is not a fractions chapter; fractions as numbers on a line are Chapter 6. Today doubling is equal groups: two of the same amount.

The fun. More muffins. Being the person who doubled.

The skill. Doubling as two equal groups. In the eight-to-ten band, $2 \times$ a quantity. Still write the equation. The kitchen motivates. It does not replace the written number.

Say: "We need two of this amount. Is that equal groups, or something else?" If they add the two numbers from the recipe card that happen to sit next to each other, they joined print instead of grouping. Pour one, pour another the same, write $2 \times$ 1 cup = 2 cups, or 2×4 muffins = 8, whatever the actual numbers are. Look. Do not guess.

5–7. Two plates of the same snack. Two scoops. Talk it.

8–10. Double a two-ingredient list. Write both sentences. A later cousin, labelled, not dumped: triple as three groups.

Named try-it: Store "three at forty" (out-of-home)

Time. Five to ten minutes of a trip you were taking anyway.

Materials. A generic price tag. Optional: coins already in a pocket. A scrap if you will write on a bench after.

Safety. Aisle courtesy. A price is a number. It is not a sermon about a food system. No public quiz of a cashier. No shoplifting a "test." No lecture to a stranger's child.

The fun. How much for three. Being the person who can tell before the register.

The skill. Equal groups. Never "altogether means add." Five-to-seven with small whole numbers — three apples, two each, how many. Eight-to-ten with a two-step: three at forty cents, then change from two dollars. Still name the type for each step.

Say: "Is this equal groups, or something else?" If they add 3 + 40 because *altogether* is in their head, they grabbed. Three groups of forty. Sketch three marks of 40 on the scrap later, if the aisle is not the place to write. The equation can wait until the bag is in the car. The type cannot wait: name it while the tag is in view.

5–7. Three bananas, or two boxes of the same thing. Whole numbers they can count.

8–10. Unit price as equal groups. A compare sitting next to it if two tags are in view: which is more, how many more — that is a compare, not equal groups. Mixing in the aisle is allowed once the type question is fluent.

Named try-it: Bakery pack / playground teams (out-of-home)

Time. Five to ten minutes.

Materials. A pack of twelve (rolls, stickers, a box of crayons already bought). Or a hypothetical, labelled: eight children and two teams, said as a story, not as a census of the playground.

Safety. Leftover named without shame. No lecture to other families. Roads. Other children. You are not putting anyone on trial for not sharing with you.

The fun. Fair packs. Being the person who can tell how many teams, or how many in a team.

The skill. Partitive versus measurement. Leftover is remainder, not "subtract the friends."

A pack of twelve, four to a bag: how many bags? Measurement. A pack of twelve shared among four people: how many each? Partitive. Same twelve. Two stories. If two are left when you pack fives, two are leftover.

Say: "Is this equal groups, or something else? Are we looking for how many groups, or how many in each?" If they subtract because someone used the word *left*, they grabbed *left* as take-away. Table 5.2 already warned that *left* fails on remainder. Name leftover. Write it.

5–7. A pack of six or eight. Talk. You write.

8–10. Remainder written. A strip: a bar of twelve, split into four equal parts. Then the equation.

Talk box

This is an illustration, not a reported family. Three bags of four crackers are on the table. The parent has not announced a definition of multiplication.

Opening question

"Is this equal groups, or something else?"

Not: "What is the vocabulary word?" Not: "What does altogether mean?" Not: "Recite the fours."

Follow-ups (pick three to five; you will not use all of them every sitting)

1. What is in each group? How many groups?
2. Show me with things, a sketch, the number line, or a number sentence.
3. Are we looking for the total, the number of groups, or the size of a group?
4. Does that answer make sense? Could there really be sixteen eggs in four cartons of twelve?
5. You may change your mind. Try it another way. Altogether showed up — does that mean add this time?

How to wait

Ask. Count a slow three in your head. Look at the groups, not at the child's face, if the silence is hard. After they stop talking, wait again before you speak. If they are mid-count or mid-reason, do not cut them off. A follow-up that arrives in half a second trains the child to hunt a word. Three seconds is a convention, not a sacrament — Stahl's think-time, from classroom teaching. Rowe's windows were science class. Use the pause. Neither is a homeschool trial of equal groups.

What a stuck silence usually means

They are hunting a keyword. Or wait-time after the question was zero, and they learned that you will fill it. Or they added 4 + 12 because *altogether* showed up. Or they think division is "the one with the little house." Or the numbers are too big. Or they offered a skip-count recitation where a type was asked for. Or they are guessing what you want. Or the question was vague. Next move: smaller numbers, the crackers, wait, point back at the groups. "What is in each bag?" is a hint. The finished product is not a hint; it is the grab. Done enough this week: a spoken sentence and a sketch. "Three groups of four. Twelve. $3 \times 4 = 12$."

For the student

This page is for you.

Equal groups means the groups are the same size. Three bags of four crackers. Two teams of five. Four cartons with twelve in each.

Multiplication is one equal-groups story: how many in all.

Division is two equal-groups stories. How many groups — that is packing. How many in each group — that is sharing out.

They are the same picture, said different ways.

It is not a times-table chant as the first job. It is not a hunt for the word *altogether*. *Altogether* sometimes sits in an adding story and sometimes sits in an equal-groups story. You have to look at the groups.

A tiny worked example

Three bags.

Four crackers in each bag.

Are the groups equal? Yes.

How many in all?

You can count 4, then 8, then 12.

You can write $3 \times 4 = 12$.

You can write $12 = 3 \times 4$, because both sides are the same amount.

Now empty the bags.

Twelve crackers. Four in each bag. How many bags?

That is $12 \div 4 = 3$.

Now three plates.

Share twelve onto three plates. How many on each?

That is $12 \div 3 = 4$.

Same crackers. Three stories.

Two tries

1. Make two groups of five, with crackers, pasta, or coins (coins stay with a grown-up if they are a choking hazard). Say how many in all. Write $2 \times 5 = 10$ if writing is easy. If writing fights the sitting, talk the sentence.
2. Take a pile of twelve. Pack them four to a cup. How many cups? Then share twelve onto three plates. How many on each? Say which story is packing and which story is sharing out.

Explain it back

Tell someone at the table, in your own words, what equal groups are. Then say one story that is *not* equal groups — eight crackers and then three more, or which pile has more. Then point at four cartons of twelve and say why 4 + 12 is the wrong picture.

Challenge

Someone hears "altogether" and always adds. Someone hears "each" and always multiplies. Someone can chant 4, 8, 12, 16 and cannot make four groups of four. What would those three moves mean? How do you look at the *groups* instead? How do you ask whether this is equal groups, or something else?

You are allowed to struggle. You may use crackers, a sketch, a number line, a number sentence. You build. You talk. If you get stuck, ask for a hint — not the finished number. Then try again.

When you talk, a sentence about the groups is enough for today. You do not have to write a paragraph. You do not have to call yourself a multiplication person. Today you make equal groups, you name the story, and you may change your mind when you look.

A picture a computer made of three bags is a scene. It is interesting. It is not the crackers.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The child adds whenever a word feels like a total, and cannot yet see groups.

4 + 12 on four cartons of twelve. 3 + 4 on three groups of four. *Altogether* is doing the work of a type. Next move: smaller groups, nearer objects. Two groups of three. No story words at all, the first day. Build. Sketch. Then a story that uses *altogether* on purpose, as a trap you will name after they try. If this is still the bottleneck after several weeks of short sits, stay here. Arrays and area will not hold on top of a child who cannot yet make two groups of three. A human who will sit with crackers and wait, not a tablet that prints $3 \times 4 = 12$, is a reasonable next step if you have tried the small-set work and the keyword grab is still the whole hour.[90]

2. The child can chant a table, or skip-count fluently, and cannot tell packing from sharing out.

The cue is still a rhythm, or a poster, or the little house. Next move: objects first, type second, symbol third. Cover any table. Finger on the bags. "How many groups?" and "how many in each?" as two different sittings. After an attempt, one incorrect example to explain ("this person wrote 75×3 because they saw *each* — what did they use?"). Slow down the chant until a sentence can be tied to a sketch. A child who can recite the fours and cannot pack a pile of twelve is not ready to skip this. Go ahead once they can build the groups, name the unknown, and write a matching sentence. A tutor is useful if recitation remains the default after a couple of weeks of daily object-first work *and* the hour has become a fight.

3. The child will not start, or every sitting ends in a shrug, and the silence after your question is a wall.

The groups are the product, and a shrug is the mind at rest — or the wait was zero, or the numbers are too big, or they think you want a table. Next move: shorter sits, six crackers, the opening question from the talk box, wait a slow three twice.

[90] Thomas P. Carpenter, Elizabeth Fennema, Penelope L. Peterson, Chi-Pang Chiang, and Megan Loef, "Using Knowledge of Children's Mathematics Thinking in Classroom Teaching: An Experimental Study," *American Educational Research Journal* 26, no. 4 (1989): 499–531. Twenty first-grade teachers randomised to a month-long workshop on children's addition/subtraction problem solving; twenty control. Experimental teachers taught problem solving more and number facts less; students exceeded controls on number facts, problem solving, reported understanding, and confidence. Student N and cell means beyond the 20/20 teacher assignment are not independently re-opened here. The portable finding: teaching from children's problem types moved both problem solving and facts in grade 1. Facts were not sacrificed. CGI equal-groups language (product unknown; measurement division; partitive division) is taken from public tables and from Fuchs et al. 2021; the Carpenter, Fennema, Franke, Levi, and Empson book *Children's Mathematics* was not independently opened as a page-level source for this chapter.

If silence is hard, you look at the bags, not at them. An authentic question is one you do not already have the sentence for. If you hear yourself answering, you have started doing the work. If they offered 3 + 4, they may be stuck on the mismatch; name it and rebuild. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that fills the silence, is the release valve.

When to slow down. Keyword grabs still the default. Recitation still standing in for type. Numbers so big the groups cannot be built. Sits so long that making never starts. Those are brakes. "Not ready" because of age is the brake this book will not use. "Not ready" because two groups of three have not yet become a type is a real brake. A Grade 3 memory goal on a map is not a reason to skip the crackers.

When to go ahead. The child can build three groups of four and write $3 \times 4 = 12$. A packing story and a sharing story have both been named, with a sketch. A keyword trap has been caught at least once, and repaired with a picture. Short sits are ordinary. Then arrays, in the eight-to-ten band, have somewhere to sit. Being "good at tables" because a child loves to chant is not a reason to skip the type.

When to get a human tutor. You have run the bags, or the share, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear 4 + 12 as a keyword grab. Outsourcing the hearing is the thing to avoid, not asking for help.

Hearing which strand is thin — chant without groups, or groups without a written sentence — is enough. Correct a wrong answer without crushing the attempt. Hearing a wrong answer, then asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The short version: the child talks and builds first. You hold the crackers and the question. A tool may explain today's idea *to you* from a named lesson — three groups of four, or page whatever of the book on the table — make extra isomorphic practice with the answer key held by you, write a short SCRIPT for *you* to say after the child has tried, offer a hint after an attempt, or help you diagnose a sketch the child already made. Crop to the paper. Do not upload a child's face. The child looks at objects. The child talks in the room.

Ages 5–7: no live open chat as the child's partner. Ages 8–10: still parent in the room. Unaided first. SCRIPT after the try. COPPA's under-13 line does apply. You hold the account.

Leave these out of the hour: a chatbot as the only partner; a tool that writes $3 \times 4 = 12$ as the child's work; "what's the answer"; a camera pointed at the page so a solution pops up for the exact problem; an unsupervised window during the attempt; a tool asked to invent how many crackers are in the bag. Look in the bag.

A language model will happily finish the product. Treat every model-supplied "as we saw, there are twelve" as untrusted until it lives in the groups. Fluent talk from a tool is performance, not the child's math. In a high-school math trial, an unguarded chatbot made practice look better and left students worse when the window was closed. That paper is high-school mathematics, not a five-to-ten trial. The kitchen-table rule it supports: the model may prepare the adult and the next problem; it may not do the child's problem.[91]

A muffin tin, twelve crackers, and a scrap of paper are tools too. Use them, then fade them.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 3 math workbook" is a publisher's scope, not a legal grade, and not a transcript line. Beast Academy, Saxon, Math-U-See, and a kitchen muffin tin can place the same child in three different rooms, because they cut the grain differently. Choose by fit. The resources chapter names those programs as options. It does not rank them. The record, when you need one, still says Mathematics or Arithmetic. It does not say a brand.

Checklist before moving on

[91] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without Guardrails Can Harm Learning: Evidence from High School Mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. Field RCT, large high school in Turkey, nearly 1,000 students, grades 9–11. GPT Base +48% on assisted practice, −17% on the closed exam; GPT Tutor +127% practice, unaided exam ≈ control. Not a 5–10 trial. The sitting condition — a tool that produces the response — is the mechanism used here.

- The child can look at a small set of equal groups and say how many in all, without needing a keyword.
- A packing story (how many groups) and a sharing story (how many in each) have both been built, sketched, and — in the eight-to-ten band — written.
- $3 \times 4 = 12$ and $12 = 3 \times 4$ are both available. The equal sign still means the same as.
- 4 + 12 on four cartons of twelve has been heard, at least once, and repaired with a picture.
- Skip-count is a bridge, not the whole of the hour.
- Leftover can be named without shame.
- In the eight-to-ten band: an array of three rows of four, turned, still twelve; a two-step store story named by type.
- Sits can be short. A sketch and a sentence are enough. You did not require a chant of all products as the identity of the week.
- You can hear a keyword grab, a house-as-definition, and a table offered as type, and you can ask a good question, without taking the crackers.

If most of that list is true, go on to fractions as numbers on a line, even if the birthday says otherwise. If the birthday says "third grade" and two groups of three are still not a type, stay. The next chapter is the number line, with equal groups still in the room as the share that will become a unit fraction.[92]

A path through elementary mathematics is the promise. A diploma is not. A percentile is not.

[92]NMAP, *Foundations for Success* (2008), Finding 12, on fractions: one key mechanism linking conceptual and procedural knowledge is representing fractions on a number line — the next chapter. Finding 8: most children acquire considerable number knowledge before kindergarten. This chapter assumes Chapter 1's cardinality and Chapter 3's equal sign are available; if they are not, stay there.

Chapter 6

Fractions as numbers on a line

Figure 13: A kitchen-table still-life from a high three-quarter view: a 0–1 number line of tape on the table with 1/2 and 3/4 marked, a folded paper strip beside it, a measuring cup in the background, and a scrap with 1/2 written. No pizza as the only object. No people. No logos.

Why this matters

Where does this fraction live on the line?

That is not a vocabulary quiz. It is not a request for a pizza drawing as the finished number. It is the question that turns three-fourths from a picture of slices into a location, the way 3 is a location and 8 is a location. A half of a cracker is a fair share. A half on a 0–1 line is a number. Both belong in the week. The share is the introduction. The line is the destination of this chapter.

Pizza is part–whole. The Institute of Education Sciences fractions guide, Recommendation 1, builds on informal sharing and proportionality, and the guide's own figure is four pizzas among eight children.[93] That recommendation is rated *minimal* evidence. The panel still wanted the share. Recommendation 2 is the load-bearing move of this chapter, rated *moderate*: help students recognize that fractions are numbers that expand the number system beyond whole numbers, and use number lines as a central representational tool from the early grades onward.[94] The National Mathematics Advisory Panel's Finding 12 is blunt in the same direction. Difficulty with fractions is pervasive and is a major obstacle to further progress, including algebra. One key mechanism linking conceptual and procedural knowledge is the ability to represent fractions on a number line.[95] A diet of only pizza trains a part of a whole. It does not train a number that can sit between 0 and 1, and past 1.

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[93]Robert Siegler, Thomas Carpenter, Francis Fennell, David Geary, James Lewis, Yukari Okamoto, Laura Thompson, and Jonathan Wray, *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*, NCEE 2010-4039 (IES, 2010), Recommendation 1 (minimal evidence): build on informal sharing and proportionality. Figure 3 in the guide is sharing four pizzas among eight children. Pizza is in the evidence base as an introduction. Access date for URLs in these notes: 3 September 2026.

[94]Siegler et al., *Developing Effective Fractions Instruction* (2010), Recommendation 2 (moderate evidence): help students recognize that fractions are numbers that expand the number system beyond whole numbers; use number lines as a central representational tool from the early grades onward. Recommendation 3 (moderate): procedures that make sense. Recommendation 4 (minimal): ratio, rate, and proportion before cross-multiplication — out of this book's lesson spine; labelled at the age-10 edge, not dumped.

[95]National Mathematics Advisory Panel, *Foundations for Success* (Washington, DC: U.S. Department of Education, 2008), Finding 12: difficulty with fractions (including decimals and percents) is pervasive and a major obstacle to further progress, including algebra. "One key mechanism linking conceptual and procedural knowledge is the ability to represent fractions on a number line." The Panel ranked proficiency with fractions as the most important foundational skill not presently developed.

A third reason sits next to those two, labelled so it does not become a kitchen promise. Siegler and colleagues found that age-10 fractions knowledge uniquely predicted high-school algebra and overall mathematics five to six years later, after other elementary skills were controlled. In the UK Birth Cohort Study, a one-standard-deviation increase in age-10 fractions was associated with +0.15 standard deviations of later algebra; in the US PSID-CDS sample, +0.17. Concurrent correlations with high-school total math were high. Whole-number division was the other unique elementary predictor. Addition, subtraction, and multiplication were weaker or not significant once those controls were in. Fractions and division were generally *not* uniquely predictive of high-school literacy.[96] This is a useful study, not a promise that every home will see the same result. It is a prediction finding, not a classroom experiment proving a teaching method. It is a reason to put 1/2 on a line this year. It is not a reason to dump unlike-denominator addition onto a five-year-old.[97]

What this idea unlocks is almost everything later that looks like "part of." Equivalence as the same point. Numbers greater than one as mixed numbers that still live on the same line. Tenths as a new partition, not a new kind of number. Decimals as the same place-value system, labelled at the age-10 edge, not as a middle-school course. If a pizza drawing still stands in for 3/4, later pages arrive as two whole numbers with a bar between them, and 1/2 + 1/3 becomes 2/5 because the child added tops and bottoms. Hearing a pizza offered as the finished number, then putting 3/4 on a 0–1 line, is worth the struggle.

You do not need to be a mathematician. You do need to hear a pizza drawing that never becomes a line as an unfinished number, not a cute picture, and to ask where this fraction lives.

[96] Robert S. Siegler, Greg J. Duncan, Pamela E. Davis-Kean, Kathryn Duckworth, Amy Claessens, Mimi Engel, Maria Ines Susperreguy, and Meichu Chen, "Early Predictors of High School Mathematics Achievement," *Psychological Science* 23, no. 7 (2012): 691–697. UK BCS N = 3,677: +0.15 SD algebra, +0.16 SD total math. US PSID-CDS N = 599: +0.17 SD algebra, +0.18 SD total math. Concurrent r(fractions, total math) = .81 (UK), .87 (US). Whole-number division the other unique elementary predictor. Not uniquely predictive of high-school literacy. Labelled prediction, not a kitchen promise. Siegler, Thompson, and Schneider (2011) *Cognitive Psychology* full PDF was not independently extracted this project; theory restated in the opened 2012 paper.

[97] NMAP, *Foundations for Success* (2008), Finding 10: conceptual understanding, fluency, and problem solving together. Finding 15: "too young" claims wrong if prerequisites are present. A six-year-old who can count and fold can place 1/2; unlike denominators wait until 1/2 is a point.

This week you can learn unit fractions and the 0–1 line well enough to hear the miss. Today the student can fold a strip in half, mark 1/2, and say where it lives.

For the parent: understand it yourself

Many adults feel rusty on why 2/4 is the same number as 1/2, and why 5/4 is past 1. That is ordinary. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The child still folds.

Everyday picture. A paper strip. 0 at one end. 1 at the other. Fold in half. The fold is 1/2. Not because a pizza was halved. Because the distance from 0 to the fold is the same as the distance from the fold to 1. Two quarter-folds land on the same point as the half-fold. That same point is 1/2 and 2/4. Pouring two quarter-cups into a half-cup is the kitchen cousin. The pour motivates. The strip is still the number.

Precise picture. A *unit fraction* is one equal part of a whole that has been partitioned into n equal parts: 1/2, 1/3, 1/4, 1/8. 3/4 is three of those fourths, which is why it lives three fourth-lengths from 0. The Common Core Grade 3 map, which is a map and not a statute, wants denominators 2, 3, 4, 6, 8, unit fractions, and the number line. Grade 4 adds 5, 10, 12, 100.[98] Size of a part is relative to the whole: 1/2 of the paint in a small bucket could be less paint than 1/3 of the paint in a larger bucket. The whole has to be named.

Fractions less than 1 do not automatically teach fractions greater than 1. 5/4 is one and one fourth. It lives past 1 on the same line. Both ranges need to be taught.[99] Equivalence is same point, not a cross-multiply trick. 1/2 and 2/4 and 4/8 occupy the same location. Tenths are a new partition of the same 0–1, not a new kind of number. Hundredths are a finer partition. Decimals as tenths and hundredths sit at the age-10 *edge* of this book, as a label, not as a dumped lesson.

Unlike-denominator addition — 1/2 + 1/3 as same-size pieces, sixths — is also an

[98] National Governors Association Center for Best Practices and Council of Chief State School Officers, *Common Core State Standards for Mathematics* (2010), 3.NF: unit fractions; number line; denominators 2, 3, 4, 6, 8. Grade 4 adds 5, 10, 12, 100. Size of a part relative to the whole: 1/2 of the paint in a small bucket could be less paint than 1/3 of the paint in a larger bucket. Map, not a homeschool statute.

[99] Siegler et al. (2012), discussing the 2011 integrated theory: conceptual knowledge of fractions less than 1 does not necessarily transfer to fractions greater than 1; both ranges need to be taught.

edge, for a child toward ten who already places unit fractions and sees equivalence as the same point. The Grade 5 map and the Panel's end-of-Grade-5 benchmark put addition and subtraction of fractions there.[100] This chapter will not write a five-year-old lesson in sixths. A parent box at the end of "done enough" names what comes next. Invert-and-multiply is not a first fraction-division move. Cross-multiplication is not a first comparison move. Recommendation 4 of the 2010 guide, ratio and rate before cross-multiply, sits after this book.[101]

The 2021 intervention guide's Recommendation 4, rated strong on 14 studies with struggling K–6 students, wants the number line as a central tool for concepts and procedures. The same panel notes that number lines are not always the best picture: multiplication of two fractions less than 1 is often cleaner as an area model.[102] This book's spine does not dump two-fraction multiplication. When a later year needs it, use the picture that fits.

Money is a terrible first model of fractions. A dime is one-tenth of a dollar, and children also see "10 cents," and the whole is easy to lose. Coins belong in Chapter 7 as measurement of value. Keep the line here.

Wrong answers you should be able to hear

1. *A pizza drawing with three of four slices shaded, offered as the finished number 3/4, with no line in the week.* Part–whole happened. The number did not. Recommendation 1 is the share. Recommendation 2 is the line. Say: "That is three of four parts. Where does 3/4 live on our strip?" Fold. Mark. The pizza can stay in the room. It cannot be the destination.

2. *1/2 + 1/3 = 2/5.* Tops added, bottoms added. Two whole numbers with a bar, not two numbers on a line. The child has not yet placed 1/2 or 1/3. Stay on

[100]NMAP, *Foundations for Success* (2008), Table 2: end of Grade 4, identify and represent fractions and decimals and compare them on a number line; end of Grade 5, addition and subtraction of fractions and decimals. The Panel noted there is no empirical research on the placement of those benchmarks; they should be interpreted flexibly. Unlike-denominator addition is an age-10 edge in this book, not a five-year-old lesson.

[101]Siegler et al., *Developing Effective Fractions Instruction* (2010), Recommendation 4. Invert-and-multiply is not a first fraction-division move in this chapter. Cross-multiplication is not a first comparison move.

[102]Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*, WWC 2021006 (IES, 2021), Recommendation 4 (strong evidence, 14 studies): use the number line. The panel notes number lines are not always the best picture; multiplication of two fractions less than 1 is often better as an area model. Population: struggling K–6, intervention, not a parent-hour randomised trial.

placement. Unlike-denominator addition is an age-10 edge, not the repair for a five-year-old. The repair this week is: where does 1/2 live? Where does 1/3 live? Which is farther from 0?

3. *"1/8 is bigger than 1/4 because 8 is bigger than 4."* The denominator was treated as a whole-number size. Eight equal parts are smaller than four equal parts of the same whole. Fold the strip. Look at the length of one eighth next to one fourth. The line does the correcting. A lecture on "bigger bottom means smaller piece" without a fold is a slogan.

4. *The child counts tick marks instead of spaces.* Three ticks past 0 named as 3/4 on a strip partitioned into fourths, because they counted the hash marks and included 0, or skipped the equal-length idea. The 2021 panel flags this as a number-line obstacle: the tick is not the unit; the space is.[103] Point at the intervals. "How many equal jumps from 0 to 1?" Then "how many jumps is 3/4?"

5. *5/4 treated as illegal, or as 5 and 4, or as less than 1 because "fractions are pieces of pizza."* Conceptual knowledge of fractions less than 1 does not necessarily transfer past 1. Extend the strip. 0, 1, 2. Four fourths make 1. Five fourths live one jump past 1. Mixed number 1 1/4 is the same point.

A sixth you will also hear: a recipe half-cup poured, enjoyed, never written, never placed. The kitchen motivated. The number never arrived. Pour, then mark 1/2 on the strip the same week.

Five-minute parent warm-up

Do this before the lesson, with a paper strip and a pencil, no child in the room.

Minute 1. Mark 0 and 1. Fold in half. Mark 1/2. Say out loud: "One half lives here. It is a number."

Minute 2. Fold in half again. Mark 1/4, 2/4, 3/4. Notice 2/4 on the same point as 1/2. That is equivalence as location.

Minute 3. Extend past 1, or imagine a second strip taped on. Mark 5/4. Say: "Past 1. One and one fourth."

Minute 4. Imagine a pizza drawing offered as 3/4. Decide, without performing,

[103] Fuchs et al. (2021), Recommendation 4, number-line obstacles as described in the guide, including treating hash marks rather than intervals as the unit.

what a scold would do. Decide what "where does this live on the line?" would do instead. You are installing the miss so you can hear it tomorrow.

Minute 5. Write the sentence you will actually say: "Where does this fraction live on the line?" Put the pencil down. That sentence is the lesson.

If you can do those five minutes, you are ready to sit down. The child folds. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version. Here is the shape scaled to fractions as numbers.

Warm-up (2–5 minutes, unaided). A whole-number location already known from Chapter 1: "Where does 3 live on this 0–10 line?" Then, once halves have been marked: "Where does 1/2 live?" No device. No pizza required.

Short model (3–7 minutes). One strip, 0 and 1, one fold, 1/2 marked, said as "one half is the number that lives in the middle of 0 and 1." You talk for a few minutes. Then you stop.

Student attempt (8–15 minutes). The child folds, marks, places. One to four placements of *today's* partition: halves; then fourths; then, in the eight-to-ten band, thirds and eighths. You wait. Struggle before rescue: ask, wait, hint, then model a *different* strip.

One good question, then wait. "Where does this fraction live on the line?" Not "what is a fraction?" Not "shade three of four." After you ask, a slow three. Stahl's think-time, labelled classroom-origin. Rowe's windows were science class. Use the pause.[104]

Mixed practice (5–10 minutes). Yesterday's equal groups next to today's half. A whole-number location next to 3/4. Mixing is choosing: is this a number on the line, or a share of a snack that we will then place?

Exit ticket (2–4 minutes, unaided). Two items. Place 1/2. Place 1, or 0, or 3/4 if fourths have been folded. Done-enough is a point you can see, or a wrong point

[104]Robert J. Stahl, "Using 'Think-Time' and 'Wait-Time' Skillfully in the Classroom," ERIC Digest ED370885 (1994). Mary Budd Rowe, NARST paper, 1972, ERIC ED061103: elementary science class. Use the pause. Neither is a homeschool fractions trial.

with a reason you can use tomorrow.

That shape is a practice you impose on whatever strip, cup, or walk is already in the week. It is not a 180-day fractions calendar, and it is not a middle-school ratio spine.

Exact wording you can say

On location:

“Where does this fraction live on the line?”

“Is 3/4 more than 1/2? Show me.”

“Is 3/4 the same point as 6/8?”

On the whole:

“What is the whole? From here to here is 1.”

“1/2 of this strip. 1/2 of a longer strip would be a different length. The number 1/2 still lives in the middle of *its* 0 and 1.”

On a pizza that showed up:

“That is three of four slices. Is that the number, or a picture of a part? How could we check on the line?”

On past 1:

“What about 5/4 — past 1? Let’s extend the strip.”

On ticks:

“We count the equal jumps, not the marks as if 0 were a jump.”

When you are about to take over:

“Your fold. I’ll wait.”

Then wait. Look at the strip, not at the child’s face, if the silence is hard.

Age-band moves: 5–7 / 8–10

5–7. Informal sharing: half of a cracker, half of a sandwich, leftover named. Stand on one half of a taped floor line. Fold a strip in half. Mark 1/2. Halves of a walk. Fourths if the fold is easy; if not, stay on halves. Talk the sentence. You may write 1/2. Pizza or a shared snack is allowed as Recommendation 1. The same week, the strip. Unlike-denominator addition is not a five-year-old lesson. Tenths are not a five-year-old lesson. Invert-and-multiply is not in the room.

8–10. Unit fractions. 0–1 partitioned into 2, 3, 4, 8. Equivalence as same point: 1/2 = 2/4 = 4/8. Mixed numbers past 1: 5/4, 3/2. Tenths as a new partition of the same line, not a new kind of number. Compare 2/3 and 3/4 by placing, not by cross-multiplying. A measuring-cup sitting still ends on the strip. Age-10 *edge*, labelled, not dumped: unlike-denominator addition as “same-size pieces” (halves and thirds as sixths) in a parent “coming next” box; tenths and hundredths as both fractions and decimals, one sitting, not a finance course. Area model if two-fraction multiplication ever arrives; it is not this week’s spine.

Not a stable skill at either band: ratio as a first comparison; negative fractions; a pizza-only diet called “fractions done.” Same faculty. Different grain.

First try-it for the student

A paper strip, 0 and 1 already marked:

fold in half.

Say: “Where does one half live on the line?”

Wait. If the child points at 1, they treated the whole as the half. Fold again, look at the two equal lengths. If they shade a pizza on the back of the strip and never mark the fold, they stayed in part–whole. Ask the location again. If they mark the midpoint, they heard the number. If they stare, model on a *different* strip — you fold, you mark 1/2, you say “in the middle of 0 and 1” — then hand them a fresh strip.

Later the same week, the diagnostic item:

a pizza drawing of 3/4, then a blank 0–1 line.

Hear a refusal to leave the pizza. Hear a mark at 3. Hear a mark at the right place. After the child has attempted, you may show a worked *incorrect* example that left the pizza as the finished work and ask what that person thought a fraction *was*. Generation first. Then the named miss.

How to fade help. First sitting: you fold together, you mark, they echo “1/2 lives here.” Second: they fold, you wait, you hint (“make two equal lengths”). Third: they fold, mark, and say the location. Fourth: a cousin — fourths, or a floor line — with the first strip closed.

When to stop talking. When you hear yourself explaining 3/4 while their strip is still unfolded. When the sit has become a lecture titled *What a Fraction Is*. One good question. Stop while they still have a fold left in them.

Practice that actually builds learning

Blocked, for a new move. The day you introduce 1/2 on the line, give a short set that is only halves: strip, floor tape, half a walk. The day you introduce fourths, only fourths, with the half still visible as a landmark. Mixing too early makes the child hunt for a picture of pizza instead of a point.

Mixed, for when to use it. Later the same week, a whole-number location next to 1/2, and an equal-groups share next to a strip. Mixing is choosing: is this a number on the line, or a pack of twelve to share that we will then place as 1/4?

One incorrect example to diagnose. A pizza drawing of 3/4 offered as the finished number, with a blank line ignored. Hear the part–whole. The repair is a better question, then the fold, then — if needed — the named location. Silence in the face of a pizza that never becomes a line is not kindness.

These are illustrations, not reported families.

Named try-it: Paper-strip fold (in-home)

Time. Ten to fifteen minutes. Then stop.

Materials. A paper strip. Pencil. 0 and 1 marked. Optional: tape, so the strip can grow past 1.

Safety. Ordinary. Scissors if you cut the strip, used by the adult or by a child who already uses scissors.

The fun. Folding. Finding that 2/4 sits on the same fold as 1/2.

The skill. 1/2 as a point. Equivalence as same point. Five-to-seven stand on the fold, or point. Eight-to-ten label 1/4, 3/4, 5/4 past 1.

Say: "Where does this fraction live on the line?" Fold in half. Mark. Fold again. Mark fourths. If 2/4 is marked in a different place from 1/2, the folds were not equal, or the child thinks different names must live in different houses. Look at the paper. The paper is the check.

5–7. Halves. Fourths if the second fold is clean. Talk. You write 1/2.

8–10. Thirds by folding as close as paper allows, or by measuring equal lengths. Eighths. 5/4 on an extended strip. Equivalence said as “same point.”

Figure 14: A 0–1 paper strip folded in half and in fourths, 1/2 and 3/4 labelled, a second strip taped past 1 with 5/4 marked. No people. No logos.

Named try-it: Recipe half-cup (in-home)

Time. Eight to twelve minutes, when you were already measuring.

Materials. A half-cup already in the drawer. Two quarter-cups if you have them. A strip on the table for after the pour.

Safety. No flame for a five-year-old. Ordinary cooking if the family already cooks. A measuring cup is a cup. It is not a nutrition sermon.

The fun. Pouring. Seeing two quarters fill a half.

The skill. Recommendation 1 informal share *and* Recommendation 2 "a half is a number." Two quarter-cups make a half-cup. Look. Do not guess. Kitchen motivates. Still place 1/2 on a 0–1 line the same week.

Say: "We need a half-cup. Where does one half live — in this cup, and on our strip?" If they pour until it "looks like enough," they estimated without a unit. Fill the half-cup. Then mark 1/2. If they refuse the strip because the cup already "is" half, the kitchen ate the number. The cup and the strip in the same sitting.

5–7. Half-cup. Half a cracker beside it. Talk.

8–10. Two quarters = one half, written 1/4 + 1/4 = 1/2, then both marks on the strip. A later cousin, labelled: a third-cup if you have one, placed as 1/3, not added to 1/2 until same-size pieces are actually in reach at the age-10 edge.

Named try-it: Walk half of the block (out-of-home)

Time. Five to ten minutes of a walk you were taking anyway.

Materials. A length you can walk: a block, a bench, a playground wall. Optional: chalk if the surface allows, or two sticks for endpoints.

Safety. Roads. Other children. No lecture to strangers. Chalk only where chalk is allowed.

The fun. Standing on the half. Being the person who finds the middle.

The skill. Fair share of a *length*, not only of a pizza. The line in the world.

Say: "Where does one half of this bench live?" Walk it. Estimate. Check by walking from the other end, or by counting equal steps. If they point at the end, they treated 1 as 1/2. If they stand near the middle, ask how they could check. Five-to-seven: stand. Eight-to-ten: "If this bench is 1, where is 3/4?" Then, at home, the same numbers on a strip, so the bench is not the only picture.

5–7. Half of the bench. Half of the walk to the corner.

8–10. Fourths of a longer wall. A playground measure that writes the number when you sit down.

Named try-it: Sharing a bakery pack (out-of-home)

Time. Five to ten minutes.

Materials. A pack of 8 or 12 already bought — rolls, crackers, stickers. The strip waiting at home.

Safety. Leftover named without shame. No lecture to other families. Aisle courtesy if you are still in the shop. A pack is a pack. It is not a sermon about a food system.

The fun. Fair shares. Being the person who can tell one-fourth of the pack.

The skill. Unit fraction of a set as Recommendation 1. Then "where does one-fourth live on our strip when we get home?" so the pack is not the destination.

Say: "Eight rolls. Share among four people. How many each? That is one-fourth of the pack as a share." If they add 8 + 4 because *altogether* is still in the room from Chapter 5, they grabbed. Equal groups, partitive, from last chapter. Then, later: mark 1/4 on the 0–1 strip, so the share and the number both exist.

5–7. Half the pack. Talk. Strip at home.

8–10. Fourths. Eighths of a pack of eight as 1 each, then 1/8 on the line as a length, which is a different picture of 1/8 than "one roll." Name the whole: the pack is 1 when we place 1/8 of the pack; the line from 0 to 1 is 1 when we place 1/8 as a length. Two wholes. Same name. That conversation is eight-to-ten work.

Talk box

This is an illustration, not a reported family. A strip with 0 and 1 sits on the table. A pizza drawing of 3/4 is also there. The parent has not announced a definition of fraction.

Opening question

"Where does this fraction live on the line?"

Not: "What is the vocabulary word?" Not: "Shade the pizza." Not: "Add the tops and bottoms."

Follow-ups (pick three to five; you will not use all of them every sitting)

1. Is 3/4 more than 1/2? Show me.
2. Show me with the strip, a sketch, a cup, or a number sentence.
3. Is 3/4 the same point as 6/8? What about 5/4 — past 1?
4. Does that answer make sense? Could 1/8 really be bigger than 1/4 on the same whole?
5. You may change your mind. A pizza drawing showed three of four slices — is that the number, or a picture of a part? How could we check on the line?

How to wait

Ask. Count a slow three in your head. Look at the strip, not at the child's face, if the silence is hard. After they stop, wait again. If they are mid-fold or mid-reason, do not cut them off. Three seconds is a convention, not a sacrament — Stahl's think-time. Rowe's windows were science class. Use the pause. Neither is a homeschool fractions trial.

What a stuck silence usually means

They added 1/2 + 1/3 as 2/5, and they can feel the mismatch. Or they think a bigger denominator is bigger. Or they never left the pizza. Or they treated the tick as the unit instead of the space. Or wait-time 1 was zero. Or the partition is too fine — eighths on a first sitting. Or they are guessing what you want. Next move: a coarser partition, the strip, wait, point at 0 and 1. "Fold in half" is a hint. The finished mark for 3/4 is not a hint; it is the grab. Done enough this week: a fold, a mark, a spoken sentence. "1/2 lives in the middle."

For the student

This page is for you.

A fraction is a number. It lives on the line, the way 3 lives on the line.

One half lives in the middle of 0 and 1.

Three fourths lives three equal fourth-jumps from 0 toward 1.

Five fourths lives past 1.

A pizza can show a part of a whole. That is a useful picture. It is not the only picture, and it is not the number by itself. After the pizza, ask where the number lives.

A tiny worked example

A paper strip.

0 at one end. 1 at the other.

Fold in half.

The fold is 1/2.

Fold in half again.

The new folds are 1/4 and 3/4.

The middle fold is still 1/2, and it is also 2/4. Same point. Two names.

If you tape another strip on, 4/4 is 1, and 5/4 is one more jump past 1.

Two tries

1. Fold a real strip. Mark 1/2. Say where it lives. If you thought it lived at 1, look at the two equal lengths and mark again.
2. Stand on a taped line on the floor, or on a bench: 0 at one end, 1 at the other. Stand on 1/2. If you are older, stand on 3/4, then check by folding a strip of the same idea when you sit down.

Write nothing if writing fights the sitting. Talk the sentence. If writing is easy, write: "1/2 lives ___."

Explain it back

Tell someone at the table, in your own words, where 1/2 lives. Then say whether 2/4 is a different number or the same point. Then point at a pizza drawing of 3/4 and say how you would check on the line.

Challenge

Someone shades a pizza and stops. Someone says 1/8 is bigger than 1/4 because 8 is bigger. Someone adds 1/2 and 1/3 and writes 2/5. What would those three moves mean? How do you look at the *line* instead? How do you ask where the fraction lives?

You are allowed to struggle. You may use a strip, a cup, a walk, a number sentence. You fold. You talk. If you get stuck, ask for a hint — not the finished mark. Then try again.

When you talk, a sentence about a point is enough for today. You do not have to add fractions with different bottoms. You do not have to call yourself a fractions person. Today you place a number, and you may change your mind when you look at the fold.

A picture a computer made of a pizza is a scene. It is interesting. It is not the strip.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The child lives in pizza, and cannot yet place a number.

Three of four slices, every time. A blank line ignored. Next move: hide the pizza for a sitting. One strip. 0 and 1. One fold. 1/2. Walk a bench. Then, if you want the share, a cracker, then back to the strip the same week. If this is still the bottleneck after several weeks of short sits, stay here. Unlike-denominator addition will not hold on top of a child who cannot yet mark 1/2. A human who will fold paper and wait, not a tablet that shades slices, is a reasonable next step if the pizza is still the whole hour.

2. The child treats two numbers with a bar as two whole numbers to add, or thinks a bigger denominator is bigger.

1/2 + 1/3 = 2/5. 1/8 > 1/4. Next move: no addition yet. Placement only. Fold 1/2 and 1/4 on the same strip. Look at the lengths. Compare by location, not by a trick. Slow down any "add fractions" page until 1/2 and 3/4 can be placed and compared. A child who can chant "find a common denominator" and cannot mark 1/2 is not ready to skip this. Go ahead on equivalence as same point once halves and fourths live on the line. A tutor is useful if the two-whole-numbers habit remains after a couple of weeks of daily strip work *and* the hour has become a fight.

3. The child counts ticks, or freezes past 1, or will not start.

The unit was the hash mark, or "fractions are less than 1," or the wait was zero, or eighths arrived on day one. Next move: coarser partition, intervals pointed at, a strip that grows past 1 with 5/4 as one extra jump. Shorter sits. The opening question. Wait a slow three twice. If silence is hard, you look at the strip, not at them. If you hear yourself marking their 3/4, you have started doing the work. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that prints a number line, is the release valve.

When to slow down. Pizza still the destination. Tops-and-bottoms still the default. Ticks still counted as the unit. Sits so long that folding never starts. Those are brakes. "Not ready" because of age is the brake this book will not use. "Not ready" because 1/2 has not yet become a point is a real brake. A Grade 5 unlike-denominator page is not a reason to skip the fold, and it is not a five-year-old lesson.

When to go ahead. The child can fold a strip, mark 1/2, and say where it lives. 3/4 can be placed, in the eight-to-ten band, and compared with 1/2. A pizza has been used *and* left for the line, at least once. Short sits are ordinary. Then equivalence as same point, and numbers past 1, have somewhere to sit. Being "good at pie charts" is not a reason to skip the line.

When to get a human tutor. You have run the strip, or the cup, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear a pizza offered as 3/4. Outsourcing the hearing is the thing to avoid, not asking for help.

Correct a wrong mark without crushing the fold. Hearing a wrong answer, then asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The child folds and talks first. You hold the strip and the question. A tool may explain today's idea *to you* from a named lesson — a 0–1 strip, or the page on the table — make extra isomorphic placements with the answer key held by you, write a short SCRIPT for *you* to say after the child has tried, offer a hint after an attempt, or help you diagnose a strip the child already marked. Crop to the paper. Do not upload a child's face.

Ages 5–7: no live open chat as the child's partner. Ages 8–10: still parent in the room. Unaided first. SCRIPT after the try. You hold the account.

Leave these out of the hour: a chatbot as the only partner; a tool that marks 3/4 as the child's work; "what's the answer"; a camera pointed at the page so a solution pops up; an unsupervised window during the attempt; a tool asked to invent whether the fold is in the middle. Look at the strip.

Fluent talk from a tool is performance, not the child's math. In a high-school math trial, an unguarded chatbot made practice look better and left students worse when the window was closed. That paper is high-school mathematics, not a five-to-ten trial. The model may prepare the adult and the next problem; it may not do the child's problem.[105]

A strip, a cup, and a bench are tools too. Use them, then fade them. A generated pizza is a scene. It is not the line.[106]

What "done enough" looks like

Placement is by skill, not birthday. A "grade 4 fractions workbook" is a publisher's scope, not a legal grade, and not a transcript line. The record still says Mathematics or Arithmetic.

Checklist before moving on

- The child can fold a strip, mark 1/2, and say where it lives.

[105] Hamsa Bastani et al., "Generative AI without Guardrails Can Harm Learning," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. High-school mathematics RCT in Turkey. Not a 5–10 trial. The sitting condition is the mechanism used here.

[106] Douglas Frye et al., *Teaching Math to Young Children*, NCEE 2014-4005 (IES, 2013), Recommendation 1 (moderate): number and operations along a progression; Recommendation 4 (minimal; panel still recommends): everyday math talk. Kitchen motivates; it does not replace the line.

- A pizza, or a shared pack, has been used as a share *and* then placed on a line the same week.
- 1/8 is not treated as bigger than 1/4 of the same whole, once fourths and eighths have been folded.
- In the eight-to-ten band: 3/4 placed; 1/2 = 2/4 as the same point; 5/4 past 1.
- Ticks are not the unit; jumps are.
- 1/2 + 1/3 = 2/5 has been heard as two whole numbers, and the child has been returned to placement rather than rushed into unlike denominators.
- Sits can be short. A fold and a sentence are enough.
- You can hear a pizza-as-number, a bigger-denominator-is-bigger, and a tops-and-bottoms add, and you can ask a good question, without taking the strip.[107]

If most of that list is true, go on to measurement, time, and money, even if the birthday says otherwise. If the birthday says "fourth grade" and 1/2 still will not live on a line, stay.

Coming next, at the age-10 edge — a parent box, not a five-year-old lesson. Unlike-denominator addition as same-size pieces: 1/2 and 1/3 as sixths, once equivalence is already a point on the line. Tenths and hundredths as both fractions and decimals, the same place-value system Chapter 2 began. Those are labels at the end of this band. They are not dumped into the student page, and they are not a reason to skip halves on a strip this week. Ratio, rate, and cross-multiply sit after this book. Invert-and-multiply is not a first move.

A path through elementary mathematics is the promise. A diploma is not. A percentile is not.

[107] Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock, *Psychological Science* 26 (2015): 1480–1488. School homework-help study, first- and second-graders, not a homeschool trial. Leave the strip in the child's hands.

Chapter 7

Measurement, time, and money

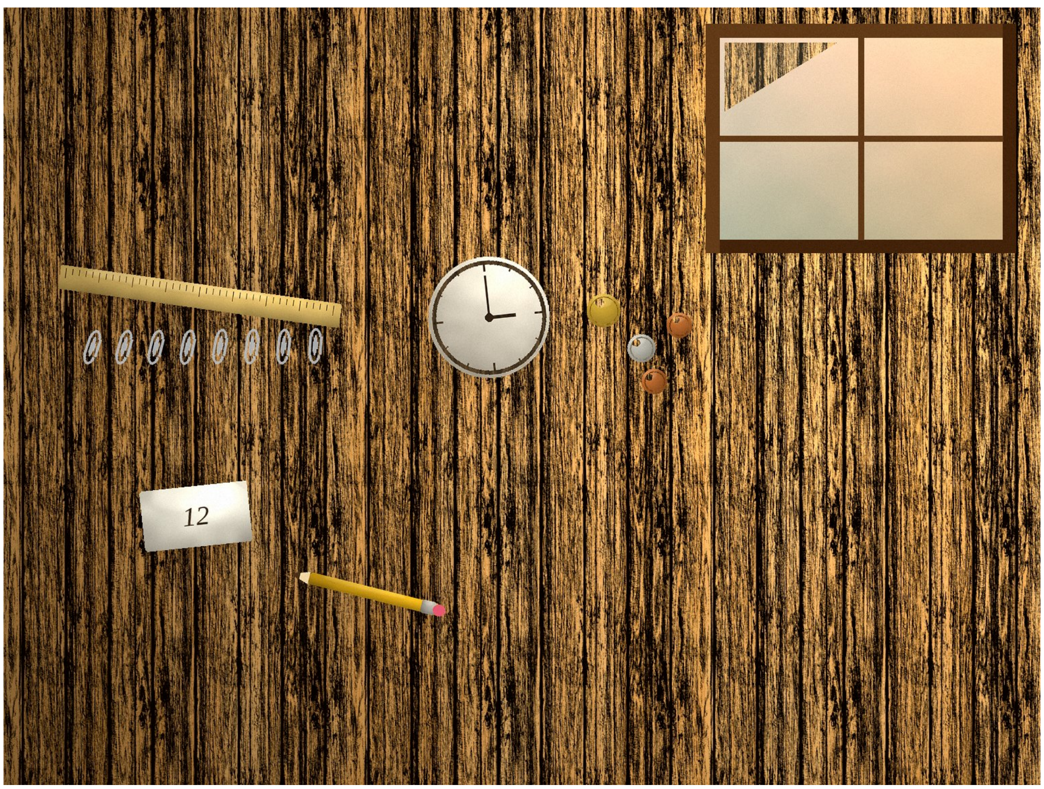

Figure 15: A kitchen-table still-life from a high three-quarter view: a ruler next to a row of paper clips along a shelf edge, a clock, generic coins, and a scrap with a handwritten number. No people. No logos. No readable package branding.

Why this matters

What are we actually measuring?

That is not a vocabulary quiz. It is not a request to read a ruler as a magic strip. It is the question that keeps length, around, covering, how-long-until, and how-much-it-is-worth from collapsing into one word, *measure*, said as if they were the same job. A shelf is longer than a book. A ribbon goes around a box. Tiles cover a floor. The clock walks from twelve to one. A dime and four pennies make fourteen cents. Kitchen and store motivate. The child still writes the number.

Measurement is direct comparison first, then informal units, then standard units. The Institute of Education Sciences early-math guide, Recommendation 2, says so, with *minimal* evidence because the studies were bundled; the panel still recommended the progression.[108] Compare two sticks. Then how many paper clips long is the shelf. Then inches, or centimetres, with the same shelf, and a sentence about why the number changed. Time is measurement, including the private homeschool headache of elapsed time — how long until lunch — which is a number-line analogue, not a keyword. Money is measurement of value. It is a terrible first model of fractions. A dime is one-tenth of a dollar, and children also see "10 cents," and the whole is easy to lose. Coins belong here. The line for 1/10 stays in Chapter 6.

What this idea unlocks is almost everything later that looks like "how much." Area as covering. Perimeter as around. The distinction matters because a bigger around does not automatically mean a bigger cover. Volume as filling, labelled at the age-10 edge. A clock that is not only "o'clock" but an interval. A price that is a number you can make with coins two ways. If the kitchen is vivid and the number is never written, later pages arrive as a ruler to guess at, a clock to freeze on, and coin names that never become an amount. Hearing "the bigger perimeter has the bigger area" as a mash to break, then measuring both, is worth the struggle.

You do not need to be a mathematician. You do need to hear a ruler read from the 1-inch mark as a magic strip, not a cute slip, and to ask what we are actually measuring.

[108] Douglas Frye, Arthur J. Baroody, Margaret Burchinal, Sharon M. Carver, Nancy C. Jordan, and Jonathan McDowell, *Teaching Math to Young Children*, NCEE 2014-4005 (IES, November 2013), Recommendation 2 (minimal evidence; panel still recommends): recognize, name, and compare shapes, then combine and separate; patterns; measurement by direct comparison, then informal and standard units; collect, organize, and represent information graphically. Ages 3–6. Access date for URLs in these notes: 3 September 2026.

This week you can learn the progression — compare, informal unit, standard unit — well enough to hear the miss. Today the student can measure a shelf in paper clips, then in inches, and say why the number changed.

For the parent: understand it yourself

Many adults feel rusty on why a smaller unit needs more iterations, and why area is not perimeter. That is ordinary. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The child still lines up the clips.

Everyday picture. Two pencils. Which is longer? Hold them together. That is direct comparison. No number yet. Then the shelf. Paper clips, end to end, no gaps, no overlaps. Eighteen clips. Then a ruler. Six inches. The shelf did not shrink. The unit got longer, so the count got smaller. Write 18 clips. Write 6 inches. The kitchen motivated. The numbers are the work.

Precise picture. *Length* is how far from here to here, iterating a unit along a path with no gaps or overlaps. *Perimeter* is length around. *Area* is covering with unit squares, or with tiles, counted. *Time* is duration, measured with a unit that the clock already iterates: hours, then half-hours, then five-minute marks, then intervals. *Money* is value, measured in cents and dollars (or the coins you actually have). The Common Core map, which is a map and not a statute, places describing and comparing attributes in Kindergarten; iterating length in Grade 1; centimetres and inches, and the inverse relation of unit size and number of units, in Grade 2; hour and half-hour in Grade 1; nearest five minutes, a.m. and p.m., and money word problems in Grade 2; time intervals, liquid volumes and masses, and area in Grade 3; converting, the area/perimeter distinction, and money as a problem context in Grade 4; volume and decimals to hundredths at the Grade 5 edge.[109] The Panel's Table 2 puts perimeter and area of triangles and trapezoids at the end of Grade 5.[110] Those are coordinates. They are not a birthday.

[109]National Governors Association Center for Best Practices and Council of Chief State School Officers, *Common Core State Standards for Mathematics* (2010). Kindergarten: describe and compare measurable attributes. Grade 1: iterate length units; tell and write time in hours and half-hours. Grade 2: measure in centimetres and inches; inverse relation of unit size and number of units; time to the nearest five minutes, a.m./p.m.; money word problems. Grade 3: time intervals; liquid volumes and masses; area. Grade 4: converting; area/perimeter distinction; money as a problem context. Grade 5 edge: volume; decimals to hundredths. Map, not a homeschool statute.

[110]National Mathematics Advisory Panel, *Foundations for Success* (2008), Table 2: end of Grade 5, perimeter and area of triangles and trapezoids. The Panel noted there is no empirical research

The inverse relation is the sentence you will actually need: if the unit is smaller, you need more of them to cover the same length. Paper clips, then inches. Steps, then a tape. That sentence is eight-to-ten work said in five-to-seven hands.

Elapsed time is measurement along a line analogue. From 2:00 to 2:30 is a half-hour jump, not a keyword hunt for *until*. This book does not claim a separate experimental literature on elapsed-time instruction; treat a specific effect size as unknown and teach the clock as a measuring tool.[111]

Money as decimal-adjacent — $1.60, the 1 as a dollar, the 6 as six tenths of a dollar — is an age-10 label, not a middle-school finance course. Five-to-seven: coin combinations as measurement of value. Eight-to-ten: change, and two ways to make fifty cents. Still write the amount. Coin names do not replace 1/10 on Chapter 6's line.

Wrong answers you should be able to hear

1. *The child starts the ruler at 1, or reads the number under the end without iterating, and calls that the length.* The ruler was a magic strip. Length is iterating a unit from 0, or from the end of the object aligned with 0. Line the end up. Count the units. Write the number. Then check with paper clips.

2. *"The bigger around has the bigger cover."* Perimeter mashed with area. A long skinny rectangle can have a large around and a small cover. A square can cover more with less around. Measure both. Write both. The mash breaks on the numbers, not on a slogan.

3. *Coin names offered as the amount, with no written number: "a dime and a nickel" as if that were fourteen cents already said.* The names are useful. The measurement is the value. Write 15c, or 15 cents. Two ways to make it, in the eight-to-ten band. Do not let "10 cents" stand in for 1/10 on last chapter's line.[112]

4. *"How long until lunch" answered by reading the hour hand as the answer, or by subtracting the clock numbers as if they were an addition story.* Elapsed

on the placement of those benchmarks; they should be interpreted flexibly.

[111] No opened IES recommendation isolates elapsed-time instruction. Specific elapsed-time effect sizes are treated as unknown in this chapter. The teaching move is time as measurement and elapsed time as a number-line analogue.

[112] Siegler et al., *Developing Effective Fractions Instruction*, NCEE 2010-4039 (IES, 2010), Recommendation 2: number line as central tool. Money is a terrible first model of fractions; coins stay in this chapter as measurement of value; 1/10 stays on Chapter 6's line.

time is an interval. From 11:00 to 12:00 is one hour if lunch is at 12. A number-line sketch of the clock, or a jump of half-hours, is the picture. A keyword *until* is not an operation.[113]

5. *Gaps and overlaps in the paper-clip line, then a count treated as exact.* The unit was not iterated. Direct comparison first if the clips will not stay. Then try again, end to end, no gaps. The number is only as good as the iteration.

A sixth you will also hear: a vivid kitchen, a poured cup, a delighted child, no numeral anywhere. The motivation did its job. The measurement did not. Pour, then write.

Five-minute parent warm-up

Do this before the lesson, with two sticks, a handful of paper clips, and a ruler, no child in the room.

Minute 1. Hold the sticks together. Say: “Which is longer? I am comparing. I do not have a number yet.”

Minute 2. Line clips along a book. Count. Write the number. Check for gaps.

Minute 3. Measure the same book in inches. Write that number. Say out loud: “Smaller unit, more of them. The book did not change.”

Minute 4. Imagine “the one with the bigger around has the bigger area.” Decide what measuring both would do. You are installing the mash so you can hear it tomorrow.

Minute 5. Write the sentence you will actually say: “What are we actually measuring?” Put the pencil down. That sentence is the lesson.

If you can do those five minutes, you are ready to sit down. The child lines up the clips. You hear.

[113] Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics*, WWC 2021006 (IES, 2021), Recommendation 5, Table 5.2: keyword cues fail; money and store stories in this chapter still name the type (join, separate, compare, equal groups), not a cue word. Recommendation 1: mix previously and newly learned material.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version. Here is the shape scaled to measurement.

Warm-up (2–5 minutes, unaided). Two lengths the child can see: “Which is longer?” Or a time they already know: “Where is the hour hand at 3:00?” No device. No worksheet dump of clocks.

Short model (3–7 minutes). One object, one unit, one written number. A shelf, paper clips, 18, said as “we iterated the clip. No gaps.” You talk for a few minutes. Then you stop.

Student attempt (8–15 minutes). The child compares, then iterates, then, if the unit is known, uses a ruler or a clock. You wait. Struggle before rescue: ask, wait, hint, then model a *different* object.

One good question, then wait. “What are we actually measuring?” Not “what does the ruler say?” After you ask, a slow three. Stahl’s think-time, labelled. Rowe’s windows were science class.[114]

Mixed practice (5–10 minutes). Yesterday’s equal groups or yesterday’s 1/2 next to today’s length. A coin amount next to a clock interval, once both exist. Mixing is choosing: length, around, covering, duration, or value?

Exit ticket (2–4 minutes, unaided). Two items. Measure a short object in informal units. Write the number. One yesterday skill beside it.

That shape is a practice you impose on whatever shelf, clock, or store is already in the week. It is not a 180-day clock-worksheet dump.

Exact wording you can say

On the job:

“What are we actually measuring?”

“Longer, or around, or covering, or how long until, or how much is it worth?”

On the unit:

[114] Robert J. Stahl, “Using ‘Think-Time’ and ‘Wait-Time’ Skillfully in the Classroom,” ERIC Digest ED370885 (1994). Mary Budd Rowe, NARST paper, 1972, ERIC ED061103: elementary science class. Use the pause. Neither is a homeschool measurement trial.

“Show me the unit.”

“If the unit is smaller, do we need more or fewer?”

“Write the number.”

On the ruler:

“Line the end up with 0. Then count the units.”

“Does that answer make sense? Could this book really be 12 inches if it is shorter than my hand span?”

On the clock:

“Where was the hand? Where is it now? How far did it travel?”

On coins:

“How much is this worth? Write it. Can we make the same amount another way?”

When you are about to take over:

“Your clips. I’ll wait.”

Then wait. Look at the shelf, not at the child’s face, if the silence is hard.

Age-band moves: 5–7 / 8–10

5–7. Direct comparison: which stick, which shoe, which path on a walk. Informal units: paper clips, steps, cubes. Hour and half-hour on a clock already on the wall. Coin combinations: pennies, nickels, dimes, a dollar as a hundred cents in talk, not as a decimal course. Write the number. Talk if writing fights. No flame. Coins are a choking hazard; count them at the table, store them after. Area as covering a small card with squares, if covering is in reach; if not, stay on length.

8–10. Standard units: inches, centimetres, then feet. Inverse relation named: smaller unit, more iterations. Nearest five minutes, then intervals — how long until, how long did it take. Area in square units. Perimeter as around, measured, written, compared with area so the mash can break. Money: change, two ways to make 50 cents, a two-step store story still named by type from Chapters 5 and 8. Age-10 edge, labelled: volume as filling a small box with cubes; money as decimal-adjacent, $1.60, without replacing Chapter 6’s tenths line.

Not a stable skill at either band: converting as a first sitting; a finance course; mean-median-mode. Same faculty. Different grain.

First try-it for the student

A shelf or a book, paper clips:

line them up, no gaps.

Say: “What are we actually measuring? How many clips long?”

Wait. If the child guesses a number without lining up, they skipped the unit. Line them up together, then hand the clips back. If they leave gaps, the iteration is the lesson. If they write a number that matches the count, they measured. Later the same week, the same object with a ruler. “Why did the number change?” Hear “the shelf got shorter.” The shelf did not. The unit did.

How to fade help. First sitting: you line up, they count. Second: they line up, you wait, you hint (“no gaps”). Third: they measure and write. Fourth: a cousin — steps on a bench, then a tape — with the clips closed.

When to stop talking. When you hear yourself reading the ruler while their clips are still in a pile. One good question. Stop while they still have a unit left to place.

Practice that actually builds learning

Blocked, for a new move. The day you introduce informal length, only clips or steps. The day you introduce the clock interval, only the clock. Mixing too early makes the child hunt for a tool instead of naming the job.

Mixed, for when to use it. Later the same week, a length next to a coin amount, and a clock next to yesterday’s fraction strip. Mixing is choosing: what are we actually measuring?

One incorrect example to diagnose. A ruler started at 1, or a bigger-perimeter-means-bigger-area mash. Hear it. Measure again. Write both numbers. Silence in the face of a mashed area is not kindness.

These are illustrations, not reported families.

Named try-it: Shelf and clips (in-home)

Time. Ten to fifteen minutes. Then stop.

Materials. Two sticks or two books. Paper clips. Then a ruler. A scrap for the numbers.

Safety. Ordinary. Clips are small; choking hazard for the youngest.

The fun. Which is longer, then how many clips, then the surprise that inches give a different count.

The skill. Five-to-seven: direct comparison, then iterate. Eight-to-ten: standard units and “smaller unit means more iterations.” Write the number.

Say: “What are we actually measuring?” Compare first. Then clips. Then inches. If the numbers differ, ask why. The object is the same.

5–7. Compare. Clips. Talk. You write.

8–10. Inches and centimetres on the same shelf. The inverse sentence said by the child.

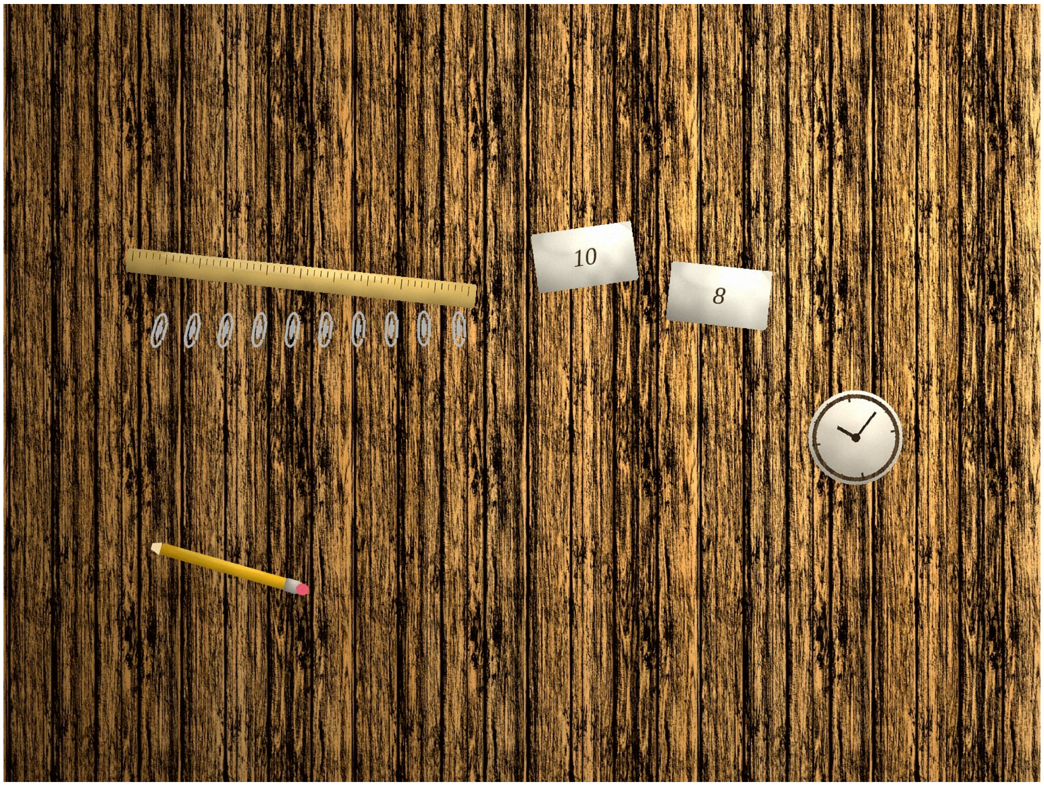

Figure 16: A ruler beside a row of paper clips along a shelf edge, two numbers written on a scrap. No people. No logos.

Named try-it: How long until lunch (in-home)

Time. Five to ten minutes, when it actually happens. Skip a fake clock drill if a real wait is coming at noon.

Materials. The clock already on the wall. Optional: a paper number line of twelve hours, or a sketch of a clock.

Safety. Ordinary.

The fun. The wait. Being the person who can tell.

The skill. Time as measurement. Elapsed time as a number-line analogue, not a keyword. Five-to-seven: hour and half-hour. Toward seven: nearest five minutes. Eight-to-ten: intervals.

Say: “What are we actually measuring? How long until lunch?” If they read the hour as the answer — “12” because lunch is at 12 — they named a time, not a duration. “It is 11:00. Lunch is 12:00. How far will the hand travel?” Jump of one hour. Write 1 hour. If *until* made them subtract clock numbers as $12 - 11$ in a story-problem grab, name the interval instead.

5–7. Half-hour until a known event.

8–10. 11:15 to 12:00 as forty-five minutes, with a sketch of jumps. A later cousin: how long did the walk take, start time and end time written.

Named try-it: Playground measure (out-of-home)

Time. Ten to fifteen minutes of a visit you were making anyway.

Materials. Steps, a stick, later a tape or a ruler you brought. A scrap for the number when you sit.

Safety. Roads. Other children. No lecture. No measuring a stranger’s property as a “test.”

The fun. How many steps long is the bench.

The skill. Length as a number line in the world. Compare two benches, or a bench and a hopscotch square. Then write the number. The walk is not a Forest School trial. It is Recommendation 4 of the early-math guide: talk about the world mathematically, *minimal* evidence, panel still recommends.[115]

Say: “What are we actually measuring?” If they count steps with giant gaps, iteration is the lesson. If they mash around-the-sandbox with covering-the-sandbox, measure both when you get home on a lid: around with a string, covering with scraps.

5–7. Steps. A stick as a unit.

[115] Frye et al., *Teaching Math to Young Children* (2013), Recommendation 4 (minimal; panel still recommends): encourage children to recognize and talk about math in everyday situations; link informal knowledge to symbols. A walk is not a Forest School trial.

8–10. A tape. Two units on the same bench. Why the number changed.

Named try-it: Store coins, making a dollar (out-of-home)

Time. Five to ten minutes of a trip you were taking.

Materials. Coins already in a pocket. A generic price tag. A scrap after, if the aisle is not the place to write.

Safety. Choking at home beforehand — count coins at the table, pocket them. Aisle courtesy. A price is a number. It is not a sermon about a food system. No public quiz of a cashier. No shoplifting a "test."

The fun. Can we make this? Two ways to make it.

The skill. Five-to-seven: coin combinations as measurement of value. Eight-to-ten: change, two ways to make 50 cents. Still write the amount. Coin names do not replace 1/10 on Chapter 6's line.

Say: "What are we actually measuring? How much is this worth?" If they name coins and never total, ask for the number. If they add 25 + 1 because they see a quarter and a 1 on a tag, they grabbed print. Look at the tag as a number. Make it with coins. Write it.

5–7. Pennies to ten. A dime as ten pennies.

8–10. Change from a dollar. Two ways to make 50 cents. $1.60 labelled as a one and six tenths, once, as an edge, then back to the cents you can count.

Talk box

This is an illustration, not a reported family. A shelf, clips, and a ruler sit on the table. A clock is on the wall. The parent has not announced a definition of measurement.

Opening question

"What are we actually measuring?"

Not: "What does the ruler say?" Not: "What is the vocabulary word?" Not: "How long until" as a keyword.

Follow-ups (pick three to five; you will not use all of them every sitting)

1. Longer, or around, or covering, or how long until, or how much is it worth?
2. Show me the unit — clips, steps, inches, minutes, cents.
3. If the unit is smaller, do we need more or fewer?
4. Write the number. Does that answer make sense?
5. You may change your mind. Try it another way. Count the clips again, or line the ruler up with 0.

How to wait

Ask. Count a slow three. Look at the shelf, the clock, or the coins, not at the child's face, if the silence is hard. After they stop, wait again. If they are mid-count, do not cut them off. Stahl's think-time. Rowe's science class. Use the pause. Neither is a homeschool measurement trial.

What a stuck silence usually means

They mashed area and perimeter. Or they read the ruler as a magic strip. Or they hunted *how long until* as a keyword. Or they used coin names instead of a number. Or wait-time 1 was zero. Or the unit is too small and the count is huge. Or they are guessing what you want. Next move: a shorter object, a bigger unit, wait, point back at the clips. "Line them up with no gaps" is a hint. The finished number is not a hint; it is the grab. Done enough this week: a count, a written number, a sentence about the unit.

For the student

This page is for you.

Measuring is using a unit, again and again, with no gaps and no overlaps, and then writing how many.

Sometimes the job is longer. Sometimes it is around. Sometimes it is covering. Sometimes it is how long until. Sometimes it is how much something is worth.

Those are different jobs. Ask which one you are in.

A kitchen and a store are good places to measure. They do not replace the number. After you pour, write. After you see a price, write.

A tiny worked example

A book on the table.

Paper clips, end to end. No gaps.

You count 12 clips.

You write 12 clips.

Then a ruler. The book is 8 inches.

The book did not get shorter. An inch is longer than a clip, so you needed fewer inches.

That is the whole trick for today: name the job, name the unit, write the number.

Two tries

1. Measure a shelf, a book, or a shoe with paper clips. Write the number. Then, if you have a ruler, measure the same thing in inches. Say why the number changed.
2. Look at a real clock. Say what time it is. Then say how long until a real event — lunch, a walk, a story. If you thought the answer was just the hour of the event, say how far the hand has to travel.

Explain it back

Tell someone at the table what you measured and what unit you used. Then say one job that is *around* and one job that is *covering*, if you have tried both. Then make an amount with coins and write the number.

Challenge

Someone starts the ruler at 1. Someone says the bigger around always has the bigger cover. Someone names “a dime and a nickel” and never writes 15. What would those three moves mean? How do you look at the *unit* instead? How do you ask what we are actually measuring?

You are allowed to struggle. You may use clips, a ruler, a clock, coins. You line up. You talk. If you get stuck, ask for a hint — not the finished number. Then try again.

When you talk, a sentence about a unit is enough for today. You do not have to convert miles. You do not have to call yourself a measurement person. Today you iterate, you write, and you may change your mind when you look again.

A picture a computer made of a ruler is a scene. It is not the shelf.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The child guesses a number, or reads a ruler as magic, and will not iterate.

A number appears with no clips lined up. Or the ruler starts at 1. Next move: hide the ruler. Two sticks. Which is longer. Then clips on a short book, you and the child both lining up, then their turn. Write the number together the first time, then their numeral. If this is still the bottleneck after several weeks of short sits, stay here. Area and intervals will not hold on top of a child who will not iterate a clip. A human who will line up clips and wait, not a tablet that prints 6 inches, is a reasonable next step if guessing is still the whole hour.

2. The child mashes jobs: around with covering, a time of day with a duration, coin names with an amount.

"Bigger perimeter, bigger area." "Lunch is 12" as how long until. "A dime" as if that were the written value. Next move: one job per sitting for a few days. Around a lid with string, covering the lid with scraps, two numbers. A clock interval with a sketch of jumps. Coins counted, then written. A child who can name inch, hour, and dime and cannot tell the jobs apart is not ready to skip this. Go ahead on standard units once informal iteration is clean. A tutor is useful if the mash remains after a couple of weeks of daily one-job work *and* the hour has become a fight.

3. The child will not start, or the kitchen was fun and no number was written, or the silence is a wall.

Motivation ate the measurement, or wait-time 1 was zero, or the object is too long. Next move: a shorter object, a bigger unit, the opening question, wait a slow three twice. Write the number as the last move of every sitting, even a two-clip sitting. If silence is hard, you look at the clips, not at them. If you hear yourself reading the ruler, you have started doing the work. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that fills the silence, is the release valve.

When to slow down. Guessing still the default. Jobs still mashed. Numbers still unwritten after a vivid pour. Sits so long that lining up never starts. Those are brakes. "Not ready" because of age is the brake this book will not use. "Not ready" because two sticks have not yet been compared is a real brake. A Grade 4 conversion page is not a reason to skip paper clips.

When to go ahead. The child can compare two lengths, iterate an informal unit, and write a number. A ruler lined up with 0 has produced a count that makes sense. A clock interval or a coin amount has been written at least once. Short sits are ordinary. Then standard units, and the inverse sentence, have somewhere to sit. Being "good at helping in the kitchen" is not a reason to skip the numeral.

When to get a human tutor. You have run the clips, or the clock, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear a mashed area and a magic ruler. Outsourcing the hearing is the thing to avoid, not asking for help.

Correct a wrong number without crushing the lining-up. Hearing a wrong answer, then asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The child measures and talks first. You hold the clips and the question. A tool may explain today's idea *to you* from a named lesson — this shelf, this clock, these coins — make extra isomorphic measures with the answer key held by you, write a short SCRIPT for *you* to say after the child has tried, offer a hint after an attempt, or help you diagnose a number the child already wrote. Crop to the paper. Do not upload a child's face.

Ages 5–7: no live open chat as the child's partner. Ages 8–10: still parent in the room. Unaided first. SCRIPT after the try. You hold the account.

Leave these out of the hour: a chatbot as the only partner; a tool that writes 6 inches as the child's work; "what's the answer"; a camera pointed at the page so a solution pops up; an unsupervised window during the attempt; a tool asked to invent how long the shelf is. Look at the shelf.

Fluent talk from a tool is performance, not the child's math. In a high-school math trial, an unguarded chatbot made practice look better and left students worse when the window was closed. That paper is high-school mathematics, not a five-to-ten trial. The model may prepare the adult and the next problem; it may not do the child's problem.[116]

Clips, a ruler, a clock, and coins are tools too. Use them, then fade the informal unit when the standard unit is accurate. A generated clock is a scene. It is not the wall.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 2 measurement workbook" is a publisher's scope, not a legal grade, and not a transcript line. The record still says Mathematics or Arithmetic.

Checklist before moving on

- The child can compare two lengths by holding them together.
- An informal unit has been iterated with no gaps, and the number written.
- A standard unit on the same object has produced a different count, and — in the eight-to-ten band — a sentence about why.
- A clock time can be read at hour or half-hour (5–7), nearest five minutes or an interval (8–10).
- A coin amount can be made and written. Coin names are not the finished measurement.
- Around and covering have been measured as two jobs, at least once, if area is in the band.
- The kitchen and the store motivated, and a number still got written.
- You can hear a magic ruler, a mashed area, and an unwritten pour, and you can ask a good question, without taking the clips.[117]

[116] Hamsa Bastani et al., "Generative AI without Guardrails Can Harm Learning," *PNAS* 122, no. 26 (2025): e2422633122. High-school mathematics RCT in Turkey. Not a 5–10 trial.

[117] Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock, *Psychological Science* 26 (2015): 1480–1488. School homework-help study. Leave the

If most of that list is true, go on to shapes, early data, and story-problem types, even if the birthday says otherwise. If the birthday says “second grade” and clips still will not line up, stay.

Coming next, at the age-10 edge — a parent box. Volume as filling a box with cubes. Money as decimal-adjacent, the same place-value idea as tenths. Converting among units once the inverse sentence is easy. Those are labels at the end of this band. They are not a five-year-old lesson, and they are not a reason to skip two sticks this week.[118]

A path through elementary mathematics is the promise. A diploma is not. A percentile is not.

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clips in the child’s hands.

[118]NMAP, *Foundations for Success* (2008), Finding 10: conceptual understanding, fluency, and problem solving together. Finding 15: “too young” claims wrong if prerequisites are present.

Chapter 8

Shapes, early data, and story-problem types

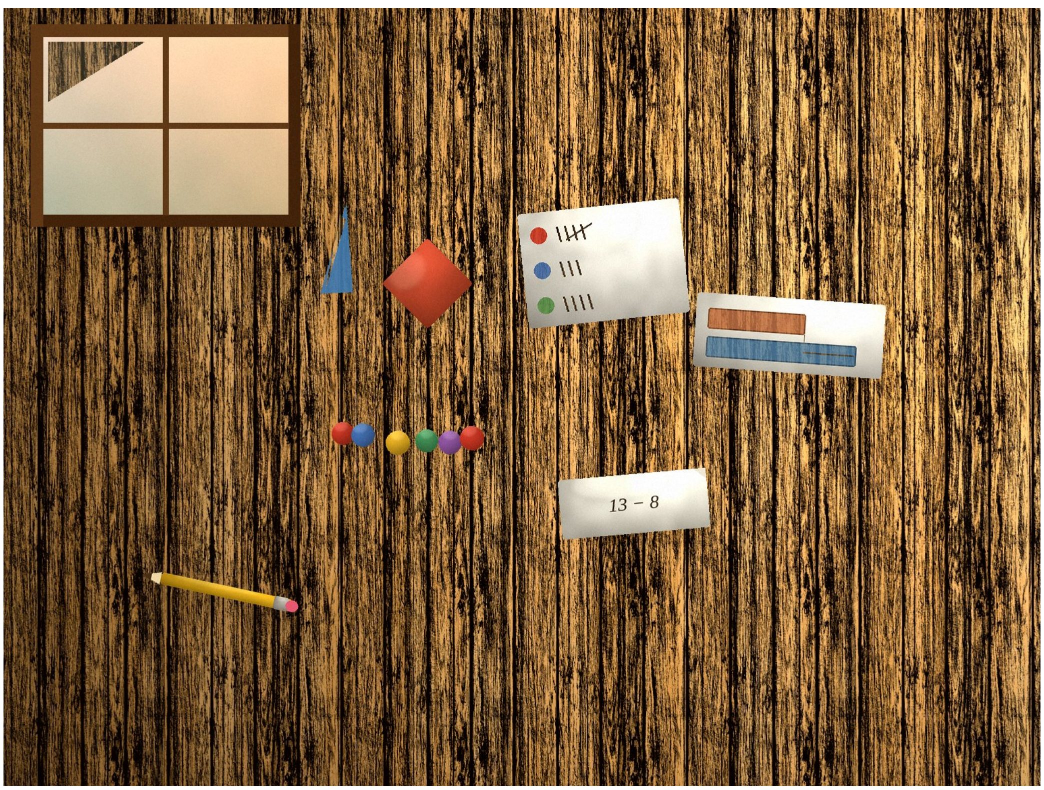

Figure 17: A kitchen-table still-life from a high three-quarter view: a skinny triangle and a square standing on a vertex, a simple three-category tally on paper, a strip diagram of a compare story, counters, and a scrap with a number sentence. No keyword poster. No people. No logos.

Why this matters

What kind of story is this?

That is not a vocabulary quiz. It is not a hunt for *altogether*, *left*, *more*, or *each*. It is the question that names a type of action or relation — join, separate, compare, equal groups — so the child can build it, sketch it, and write it. Four cartons of twelve eggs *altogether* is equal groups, not addition. Maya has eight, Jordan has thirteen, how many more is a compare, not a cue to add because *more* showed up. Geometry is real content in the same chapter, not a Friday extra. A triangle is not only the prototype pointing up. Early data is a tally and a picture graph, not inferential statistics. The hour that can name a type, name a skinny triangle, and read a bar is the hour this book has been building toward.

Word problems are types, not keyword lists. The Institute of Education Sciences 2021 intervention guide, Recommendation 5, rated strong on 18 studies, wants students to identify word-problem types that share an action or event. Identifying a type is different from grabbing an operation. It is not useful to glue a type to an operation.[119] Table 5.2 of that guide is the trap you already heard in Chapter 5: *altogether* fails on equal groups; *more* fails on start-unknown; *left* fails on remainder; *each* fails on partitive division. This chapter will not replace one poster of cue words with a better poster of cue words. Zero keyword lists as the method. The method is the type, the sketch, the sentence.

A second reason sits next to the first. Carpenter, Fennema, Peterson, Chiang, and Loef randomised twenty first-grade teachers to a month-long workshop on children's addition and subtraction problem solving, and twenty to control. Experimental teachers taught problem solving more and number facts less. Their students exceeded controls on number facts, on problem solving, on reported understanding, and on confidence.[120] Teaching from types moved both problems *and* facts.

[119]Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*, WWC 2021006 (IES, 2021), Recommendation 5 (strong evidence, 18 studies). Identifying a problem type is different from determining the operation; it is not useful to associate a type with an operation. Table 5.2: *altogether / more / left / each / share / double* fail as operation cues. Named types: Change, Combine, Compare, Equal Groups. Population: struggling K–6 intervention, not a parent-hour randomised trial. Access date for URLs in these notes: 3 September 2026. This chapter uses types as the method. It does not use a keyword list as the method.

[120]Thomas P. Carpenter, Elizabeth Fennema, Penelope L. Peterson, Chi-Pang Chiang, and Megan Loef, "Using Knowledge of Children's Mathematics Thinking in Classroom Teaching: An Experimental Study," *American Educational Research Journal* 26, no. 4 (1989): 499–531. Twenty

Facts were not sacrificed. This is a useful study, not a promise that every home will see the same result. It was teacher professional development in grade 1, not a parent-hour trial. The taxonomy is still usable at the table.

Geometry is not extras. The early-math guide's Recommendation 2, *minimal* evidence, panel still recommends, wants children to recognize, name, and compare shapes, then combine and separate them.[121] The National Mathematics Advisory Panel: early exposure to shape names "is not sufficient." Evidence on geoboards and dynamic software is "tenuous at best." Young children appear to possess at least an implicit understanding of basic Euclidean facets; formal instruction is still necessary.[122] A square on a vertex is still a square. A skinny triangle is still a triangle. Prototype-only shapes — the triangle that only counts if it points up, the square that only counts if it sits flat — are refused in this chapter.

What this idea unlocks is a durable classification you will keep using after the book is closed: join, separate, compare, equal groups, including two-step in the eight-to-ten band; properties of shapes, not pictures of shapes; a tally that becomes a bar. If "altogether means add" is still the method, later pages arrive as word hunts. If geometry is still a colouring page of perfect triangles, later properties have nowhere to sit. Hearing a keyword grab, then asking the type, is worth the struggle.

You do not need to be a mathematician. You do need to hear "altogether" as a trap, and to ask what kind of story this is, without taking the pencil.

This week you can learn the four types well enough to hear the miss, and a triangle

first-grade teachers randomised to a month-long workshop; twenty control. Experimental teachers taught problem solving more and number facts less; students exceeded controls on number facts, problem solving, reported understanding, and confidence. Student N and cell means beyond the 20/20 teacher assignment are not independently re-opened here. Teacher professional development, grade 1, school — not a parent-hour trial. Portable finding: types moved problems and facts. Facts were not sacrificed. CGI book page-level tables (*Children's Mathematics*) were not independently opened; type language from CCSS-M, Fuchs et al. 2021, and public CGI tables.

[121] Douglas Frye et al., *Teaching Math to Young Children*, NCEE 2014-4005 (IES, 2013), Recommendation 2 (minimal evidence; panel still recommends): recognize, name, compare shapes, then combine and separate; collect, organize, and represent information graphically.

[122] National Mathematics Advisory Panel, *Foundations for Success* (2008). Early exposure to shape names is not sufficient; evidence on geoboards and dynamic software is tenuous at best. In contrast to Piagetian theory, young children appear to possess at least an implicit understanding of basic facets of Euclidean concepts; formal instruction is still necessary. Finding 15: "too young" claims wrong if prerequisites are present. Finding 10: conceptual understanding, fluency, and problem solving together.

that is still a triangle when it is skinny. Today the student can look at a story, say join, separate, compare, or equal groups, and show it two ways.

For the parent: understand it yourself

Many adults feel rusty on why a compare is not a join, and why a diamond on a page might be a square. That is ordinary. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The child still builds.

Everyday picture. Eight grapes on a plate. Three more arrive. How many now? That is a *join* (add-to), result unknown. Eight grapes, three eaten. How many now? That is a *separate* (take-from), result unknown. Eight red grapes and five green. How many grapes? That is a *put-together* / combine, total unknown. Maya has eight. Jordan has thirteen. How many more does Jordan have? That is a *compare*, difference unknown. Three bags of four. How many? That is *equal groups*, product unknown. Same fruit. Five stories. The cover story can change. The type cannot be read off a single word.

Precise picture. The Common Core Grade 1 map already uses this vocabulary: add-to, take-from, put-together, take-apart, compare.[123] The 2021 guide names Change (join or separate), Combine, Compare, and Equal Groups. The unknown can sit in different slots: result, change, start, difference, product, number of groups, size of a group. "There were some. Three more came. Now there are eleven. How many were there at first?" is a join with the start unknown. It is harder. It is still a join. *More* does not mean add. The start is hiding.

A strip diagram — a bar for one amount, a bar for the other, the difference marked — is a picture the grades 4–8 problem-solving guide rates as a strong recommendation for visual representations.[124] On-age for eight-to-ten. For five-to-seven, counters and a sketch are the honest picture. One visual type at a time. Convert

[123]National Governors Association Center for Best Practices and Council of Chief State School Officers, *Common Core State Standards for Mathematics* (2010). Grade 1 critical area: add-to, take-from, put-together, take-apart, compare. Kindergarten geometry: squares, triangles, circles, rectangles, hexagons in varied sizes and orientations; cubes, cones, cylinders, spheres. Grade 2: sides and angles; picture and bar graphs; line plots in whole-number units. Grade 3: categories of shapes; scaled graphs. Grade 4: parallel, perpendicular, symmetry. Grade 5 edge: coordinate plane Quadrant I — map, not a graphing course. Data: K classify and count; Grade 1 up to three categories; Grades 4–5 line plots with fractions. Map, not a homeschool statute.

[124]John Woodward et al., *Improving Mathematical Problem Solving in Grades 4 Through 8*, NCEE 2012-4055 (IES, 2012; revised 2018), Recommendation 3 (strong): visual representations,

the picture into a number sentence. The equal sign still means the same as: 8 + 3 = 11 and 11 = 8 + 3.

Shapes: a triangle has three sides that are segments, three vertices. It need not be equilateral. It need not point up. A square has four equal sides and four right angles; standing on a vertex does not retire those properties. A rectangle has four right angles; a square is a special rectangle. In the eight-to-ten band, categories: rhombus is a quadrilateral; parallel; perpendicular; a line of symmetry. Kindergarten on the map already wants squares, triangles, circles, rectangles, hexagons in varied sizes and orientations, and cubes, cones, cylinders, spheres.[125] Prototype-only is the miss: "that is not a triangle, it is too skinny." It is a triangle. Ask how many sides.

Early data: classify and count; two or three categories; a picture graph or a bar; a line plot in whole-number units, then, later, with unit fractions from Chapter 6. Mean, median, and mode as a named trio is an eight-to-ten *label* at most, not a statistics course. A weather tally and a "who likes pears" chart are the object.

Wrong answers you should be able to hear

1. *4 + 12 = 16 on four cartons of twelve altogether.* Keyword grab. Equal groups, product unknown. Draw four circles of twelve. Write 4×12. The word *altogether* is named as a trap, once. It is not replaced with a list of safer words.

2. *The child adds because they see more: "Jordan has 5 more than Maya. Maya has 8. How many does Jordan have?" they write 5 + 8 correctly this time — then, the next day, "Maya has 8 more than Jordan" they add again when the unknown is the smaller amount. More* is not an operation. Compare stories sometimes add and sometimes subtract. Ask what we know, what is hiding, which amount is larger.

3. "*That is not a triangle. Triangles point up.*" Prototype-only. Rotate the paper. Count sides. A skinny triangle cut from paper is still three sides. A square on a vertex is still a square. The name follows the properties, not the poster.

including strip diagrams. Recommendation 2 (monitor and reflect) moderate. Floor grade 4, on-age for 8–10.

[125] CCSS-M Kindergarten geometry, as in note 123. Prototype-only instruction is refused in this chapter as a teaching move; Clements, Wilson, and Sarama (2004) shape-composition full paper page-level percentages were not independently extracted this project.

4. *A bar graph read as "the tall one wins," with no sentence about what the bar actually shows.* Early data mashed into a contest. Ask: what does this bar show? How many? Which category has more? How many more — a compare, sitting inside the chart.

5. *A two-step story treated as one grab: three apples at forty cents, then how much change from two dollars, answered as 3 + 40 + 2 or as a single keyword.* Two types in sequence. Equal groups, then a separate or a compare. Name each step. Eight-to-ten work. Five-to-seven can do one step.

A sixth you will also hear: a fluent recitation of "join, separate, compare, equal groups" with no sketch. The names are not the type. Build it.

Five-minute parent warm-up

Do this before the lesson, with counters and a scrap, no child in the room.

Minute 1. Write four tiny stories: join, separate, compare, equal groups. Do not use *altogether* as a cue. Build each with counters.

Minute 2. Take "4 cartons of 12 altogether." Write 16 as the miss. Draw four groups. Write 4×12.

Minute 3. Draw a skinny triangle and a square on a vertex. Say out loud: "Still a triangle. Still a square. I am counting sides and angles, not matching a poster."

Minute 4. Imagine the child saying "that is not a triangle." Decide what counting sides would do. You are installing the miss.

Minute 5. Write the sentence you will actually say: "What kind of story is this?" Put the pencil down. That sentence is the lesson.

If you can do those five minutes, you are ready to sit down. The child builds. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version. Here is the shape scaled to types, shapes, and a tally.

Warm-up (2–5 minutes, unaided). One known fact. One tiny story already typed yesterday: "Was that a join or a compare?" No keyword card. No device.

Short model (3–7 minutes). One story, one picture, one sentence. Or one skinny triangle, three sides counted. You talk for a few minutes. Then you stop.

Student attempt (8–15 minutes). Two or three items of *today's* type, or a shape hunt with a distractor, or a three-category tally. The child builds. You wait.

One good question, then wait. "What kind of story is this?" For shapes: "How many sides? Is this still a triangle?" For data: "What does this bar actually show?" A slow three. Stahl's think-time. Rowe's science class.[126]

Mixed practice (5–10 minutes). Yesterday's type next to today's. A compare next to equal groups. A square on a vertex next to a skinny triangle. Mixing is the life of the habit in miniature: last week's type does not leave the room.

Exit ticket (2–4 minutes, unaided). One story to type and show two ways. Or one shape to name by a property. One yesterday skill.

That shape is a practice you impose on whatever story, walk, or chart is already in the week. It is not a 180-day keyword calendar.

Exact wording you can say

On type:

"What kind of story is this?"

"What happened? What do we know? What are we trying to find?"

"Show me with things, a sketch, a strip, the number line, or a number sentence."

"Does that answer make sense?"

"You used *altogether* as add — is this equal groups?"

On shapes:

"How many sides? Is every side the same?"

"If I turn it, is it still a square?"

"A skinny one can still be a triangle. Count."

On data:

[126]Robert J. Stahl, ERIC Digest ED370885 (1994). Mary Budd Rowe, NARST 1972, ERIC ED061103, elementary science class. Use the pause. Neither is a homeschool types trial.

"What does this bar actually show?"

"Which has more? How many more?" — a compare inside the chart.

When you are about to take over:

"Your sketch. I'll wait."

Then wait. Look at the counters, the shape, or the tally, not at the child's face, if the silence is hard.

Age-band moves: 5–7 / 8–10

5–7. Join, separate, put-together, compare, with small numbers and the unknown mostly in the result. Equal groups as sharing and packing from Chapter 5, small. Shapes: name and compose; variants and distractors already in the room — skinny triangle, square on a vertex, a rectangle that is not a square. Data: two or three categories, a tally, a picture graph. Talk. You may write the sentence. One step, not two.

8–10. The same types with the unknown in other slots: start unknown, change unknown, compare with the larger or the smaller hiding. Equal groups including both divisions and remainder. Two-step stories: name each type. A strip for compare. Shapes: sides, angles, categories (a square is a rectangle; a rhombus is a quadrilateral); parallel and perpendicular as looks-and-checks, not a geometry course; one line of symmetry. Data: a scaled bar or a line plot, including unit fractions if Chapter 6 is alive. Coordinate plane Quadrant I is a map coordinate at the age-10 edge, not a graphing course.[127] Mean-median-mode, if named at all, is a label on a small set, not a trio to chant.

Not a stable skill at either band: a keyword list; prototype-only shapes; inferential statistics. Same faculty. Different grain.

First try-it for the student

A small story, counters in reach, no cue-word card:

Maya has 8. Three more come. How many now?

Say: "What kind of story is this?"

[127] CCSS-M Grade 5 geometry: coordinate plane first quadrant as a map coordinate. Not a graphing course in this book.

Wait. If they say "add because more," they grabbed. Build 8, join 3, then name join. If they say join and build it, they heard the type. Later the same week, the diagnostic: four cartons of twelve altogether. Hear 16. Repair with four groups.

How to fade help. First sitting: you name the type, they build. Second: they name, you wait, you hint ("what happened?"). Third: they name, sketch, write. Fourth: a cousin type mixed in.

When to stop talking. When you hear yourself classifying while their counters are idle. One good question. Stop while they still have a type left to try.

Practice that actually builds learning

Blocked, for a new move. The day you introduce compare, only compares. The day you introduce a skinny triangle, a handful of triangles that do not look like the poster. Mixing too early makes the child hunt a word or a prototype.

Mixed, for when to use it. Later the same week, all four types in one short set. A shape hunt that includes a distractor. Mixing is the habit: last week's type next to this week's.

One incorrect example to diagnose. 4 + 12 on four cartons of twelve altogether. Or "not a triangle" on a skinny three-sided figure. Hear it. Ask the type, or count the sides. Silence in the face of a keyword error is not kindness. A keyword list as the repair is not this book's method.

These are illustrations, not reported families.

Named try-it: Shape hunt with distractors (in-home)

Time. Eight to twelve minutes. Then stop.

Materials. Lids, a square napkin stood on a vertex, a skinny triangle cut from paper, a circle (a lid), a rectangle that is not a square. A scrap for a tally of what you found.

Safety. Ordinary. Scissors used by the adult or by a child who already uses them.

The fun. "Is this still a triangle?" Being the person who caught the skinny one.

The skill. Property, not picture. Five-to-seven: name and compose (two triangles make a something). Eight-to-ten: sides, angles, categories.

Say: “How many sides? If I turn it, is it still a square?” If they refuse the skinny triangle, count sides together. The poster in their head is the miss. The paper is the check.

5–7. Name. Compose with two or three cut shapes.

8–10. Why a square is a rectangle. A rhombus cut from paper, four equal sides, not necessarily right angles.

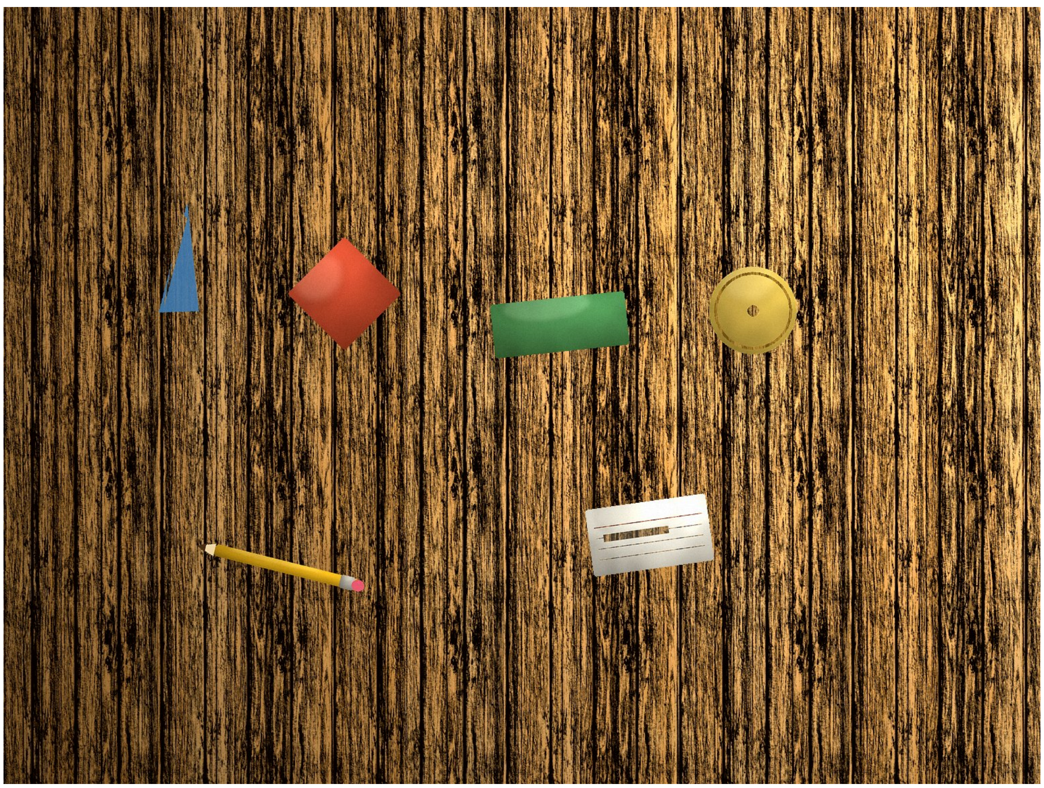

Figure 18: A skinny triangle, a square on a vertex, a rectangle that is not a square, and a lid-circle on a table. No people. No logos.

Named try-it: House tally (in-home)

Time. Ten to fifteen minutes across a few days, then one sit to make the chart.

Materials. Paper. A week of weather, or "who in this house likes pears, apples, both, neither." No body talk. No ranking people.

Safety. Ordinary. Categories about objects and weather, not about bodies.

The fun. The chart. Being the person who can tell which bar is taller and what that *means*.

The skill. Early data. Five-to-seven: tally and picture graph, two or three categories. Eight-to-ten: a scaled bar or a line plot. Mean-median-mode only as a label if you name it at all.

Say: "What does this bar actually show?" If they say "pears win," they mashed a chart into a contest. How many? How many more — a compare.

5–7. Tally marks. A picture for each vote.

8–10. One picture stands for two. A line plot of how many books on each shelf, whole numbers, or halves if Chapter 6 is alive.

Named try-it: Walk shapes (out-of-home)

Time. Five to fifteen minutes of a walk you were taking anyway.

Materials. Signs, windows, pavement rectangles, a yield sign's triangle, a square tile on a vertex if the pavement turns it.

Safety. Roads. Ordinary looking. No lecture to strangers. No "gotcha" quiz of a shop window.

The fun. Finding a skinny triangle in a roof or a sign.

The skill. Geometry in the world. Still not a Forest School trial. Frye Recommendation 2, in the wild.[128]

[128] Frye et al. (2013), Recommendation 2 and Recommendation 4 (minimal; panel still recommends): everyday math talk. A walk is not a geometry randomised trial.

Say: "How many sides? Is that still a rectangle when the window is long and thin?" If they only name the prototype they saw in a book, point at the variant. At home, cut a matching skinny triangle so the walk becomes a property, not a sightseeing badge.

5–7. Name three shapes, including one that does not look like the poster.

8–10. Parallel edges on a window. A right angle in a curb, checked with a card corner.

Named try-it: Store story (out-of-home)

Time. Five to ten minutes of a trip you were taking.

Materials. A price, a pack, a "3 more than" you can honestly see (three more apples in one pile than another). A scrap after.

Safety. Aisle courtesy. A price is a number. It is not a sermon about a food system. No public quiz of a cashier. No lecture to other families.

The fun. What kind of story is this? Being the person who can tell.

The skill. Join, separate, compare, equal groups — never a keyword. Eight-to-ten: two-step; a strip for compare.

A hypothetical, labelled, if the aisle does not offer a clean story: an eight-year-old sees "how many eggs altogether" and adds $4 + 12$. The adult asks what kind of problem (equal groups), draws four circles of twelve, writes 4×12. The keyword is named as a trap, once, without a sermon, and without a list of replacement words.

Say: "What kind of story is this?" If they grab *altogether*, ask whether the groups are equal. If they grab *more*, ask which amount is larger and what is hiding.

5–7. One step. Small whole numbers.

8–10. Two steps. Strip after, on a bench or in the car.

Talk box

This is an illustration, not a reported family. A story about four cartons of eggs sits next to a skinny triangle. The parent has not announced a definition of type, and has not put a keyword list on the table.

Opening question

"What kind of story is this?"

Not: "What does altogether mean?" Not: "What is the vocabulary word?" Not: "Which operation?" as the first question.

Join, separate, compare, or equal groups. Never a keyword as the method.

Follow-ups (pick three to five; you will not use all of them every sitting)

1. What happened? What do we know? What are we trying to find?
2. Show me with things, a sketch, a strip, the number line, or a number sentence.
3. Is this a join, a separate, a compare, or equal groups? (For shapes: how many sides? Is every side the same? For data: what does this bar actually show?)
4. Does that answer make sense? Could there really be sixteen eggs in four cartons of twelve?
5. You may change your mind. You used *altogether* as add — is this equal groups? Try it another way.

How to wait

Ask. Count a slow three. Look at the counters, the shape, or the tally, not at the child's face, if the silence is hard. After they stop, wait again. If they are mid-reason, do not cut them off. Stahl's think-time. Rowe's science class. Use the pause. Neither is a homeschool types trial.

What a stuck silence usually means

They are hunting a keyword. Or they only accept the prototype shape. Or they mashed the graph's categories. Or wait-time 1 was zero. Or the numbers are too big. Or they recited the four type-names with no sketch. Or they are guessing what you want. Next move: smaller numbers, the objects, wait, point back at the story. "What happened first?" is a hint. The finished operation is not a hint; it is the grab.

Done enough this week: a named type, a sketch, a sentence. “This is equal groups. Four groups of twelve.”

For the student

This page is for you.

Stories in math are kinds of action.

A *join* puts more with what you had.

A *separate* takes some away.

A *compare* asks how many more, or how many fewer, or which is larger.

Equal groups is the same size group, more than once — how many in all, how many groups, or how many in each.

Those names are not magic words. You still have to look at what happened.

A triangle is three sides. It can be skinny. It can lean. A square is still a square if it stands on a point.

A chart is a way to show how many in each category. The tall bar is not a winner. It is a how-many.

A tiny worked example

Maya has 8 grapes.

Jordan has 13.

How many more does Jordan have?

What kind of story is this?

We know two amounts. We want the difference. That is a compare.

Build 13. Build 8. Match them. The extras are 5.

Write $13 - 8 = 5$. Write $8 + 5 = 13$, because both sides can name the same relationship.

If the story had been four bags of 8 grapes, that would have been equal groups, not a compare. *Altogether* might have shown up. It would not have meant add.

Two tries

1. Listen to a real story at the table — snack, toys, a walk. Say join, separate, compare, or equal groups. Show it with things and with a sketch.
2. Find a triangle that does not point up, or a square that stands on a point — a napkin, a sign, a cut piece of paper. Count the sides. Say why it still has its name.

Explain it back

Tell someone at the table the four kinds of story, in your own words, with one example each. Then say why a skinny triangle is still a triangle. Then point at a tally or a bar and say what it actually shows.

Challenge

Someone hears *altogether* and always adds. Someone says a skinny three-sided shape is "not a real triangle." Someone looks at a bar graph and says "pears win" without saying how many. What would those three moves mean? How do you look at the *type*, the *sides*, or the *how-many* instead? How do you ask what kind of story this is?

You are allowed to struggle. You may use counters, a sketch, a strip, a number sentence. You build. You talk. If you get stuck, ask for a hint — not the finished number. Then try again.

When you talk, a named type and a sketch are enough for today. You do not have to keep a list of cue words. You do not have to call yourself a problem-solving person. Today you name the story, you show it two ways, and you may change your mind when you look.

A picture a computer made of a perfect triangle is a scene. It is not the skinny one on the table.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The child hunts a word, and cannot yet name a type.

Altogether means add. *Left* means subtract. *More* means add. Next move: take the words off the table. A story with no cue words at all: "8 on the plate. 3 more put on. How many now?" Build. Then a story that uses *altogether* on purpose, after they try, so the trap can be named once. If this is still the bottleneck after several weeks of short sits, stay here. Two-step stories will not hold on top of a keyword hunt. A human who will sit with counters and wait, not a worksheet of cue words, is a reasonable next step if the hunt is still the whole hour.[129] A keyword list is not the repair.[130]

2. The child only accepts prototype shapes, or reads a graph as a contest.

"Not a triangle." "Pears win." Next move: variants on the table every sitting this week. Skinny, rotated, square on a vertex. Count sides. For data: one chart, the question "how many," then "how many more" as a compare. Slow down any "geometry vocabulary quiz" until a skinny triangle can keep its name. A child who can chant hexagon and cannot count sides on a weird one is not ready to skip this. Go ahead on categories once properties hold. A tutor is useful if prototype-only remains after a couple of weeks of daily variants *and* the hour has become a fight.

3. The child recites the four type-names, or freezes, and cannot start a sketch.

The names were the performance, or wait-time 1 was zero, or the numbers are too big. Next move: smaller numbers, objects first, names second. Shorter sits. The opening question. Wait a slow three twice. If silence is hard, you look at the counters, not at them. If you hear yourself naming the type while their hands are still, you have started doing the work. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that classifies the story, is the release valve.

When to slow down. Keyword grabs still the default. Prototype-only still the default. Recitation still standing in for a sketch. Sits so long that building never starts. Those are brakes. "Not ready" because of age is the brake this book will

[129] NMAP, *Foundations for Success* (2008), Finding 23: high-quality research does not support instruction that is entirely student-centered or entirely teacher-directed. Finding 27: explicit instruction has consistently positive effects for students with mathematical difficulties — "this does not mean that all of a student's mathematics instruction should be delivered in an explicit fashion." Sequence in this chapter: child attempts; adult waits; if stuck or systematically wrong, adult models one worked example, then hands a similar item back.

[130] Russell Gersten et al., *Assisting Students Struggling with Mathematics: Response to Intervention*, NCEE 2009-4060 (IES, 2009), Recommendation 4 (strong): instruction on word problems based on common underlying structures.

not use. “Not ready” because a join and a compare have not yet come apart is a real brake. A Grade 4 two-step page is not a reason to skip eight grapes.

When to go ahead. The child can name join, separate, compare, or equal groups on a small story and show it two ways. A skinny triangle has kept its name. A tally has produced a how-many sentence. Short sits are ordinary. Then two-step, and categories of shapes, have somewhere to sit. Being “good at word problems” because a child hunts words quickly is not a reason to skip the type.

When to get a human tutor. You have run the types, or the variants, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear a keyword grab and a prototype refusal. Outsourcing the hearing is the thing to avoid, not asking for help.

Correct a wrong type without crushing the attempt. Hearing a wrong answer, then asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The child talks and builds first. You hold the story and the question. A tool may explain today’s idea *to you* from a named lesson — this compare, this skinny triangle, this tally — make extra isomorphic stories with the answer key held by you, write a short SCRIPT for *you* to say after the child has tried, offer a hint after an attempt, or help you diagnose a sketch the child already made. Crop to the paper. Do not upload a child’s face.

Ages 5–7: no live open chat as the child’s partner. Ages 8–10: still parent in the room. Unaided first. SCRIPT after the try. You hold the account.

Leave these out of the hour: a chatbot as the only partner; a tool that names the type as the child’s work; “what’s the answer”; a camera pointed at the page so a solution pops up; an unsupervised window during the attempt; a tool asked to invent how many grapes are on the plate. Look at the plate. A tool asked to generate a keyword list is the wrong object for this chapter.

Fluent talk from a tool is performance, not the child's math. In a high-school math trial, an unguarded chatbot made practice look better and left students worse when the window was closed. That paper is high-school mathematics, not a five-to-ten trial. The model may prepare the adult and the next problem; it may not do the child's problem.[131]

Counters, a skinny triangle, a tally, and a scrap of paper are tools too. Use them, then fade the counters when the sketch carries the type. A generated perfect triangle is a scene. It is not the variant on the table.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 2 math workbook" is a publisher's scope, not a legal grade, and not a transcript line. The record still says Mathematics or Arithmetic.

Checklist before moving on

- The child can look at a small story and say join, separate, compare, or equal groups, without a keyword list.
- The type has been shown two ways: things and a sketch, or a sketch and a number sentence.
- 4 + 12 on four cartons of twelve has been heard and repaired by type, at least once.
- A skinny triangle, or a square on a vertex, has kept its name by a property.
- A tally or a simple graph has produced a how-many sentence, not only a "winner."
- In the eight-to-ten band: a two-step story named by steps; a strip for a compare; a category (a square is a rectangle).
- Sits can be short. A named type and a sketch are enough.
- You can hear a keyword grab, a prototype refusal, and a recitation of type-names with no sketch, and you can ask a good question, without taking the counters.[132]

If most of that list is true, you have a path through the eight teaching chapters of

[131]Hamsa Bastani et al., *PNAS* 122, no. 26 (2025): e2422633122. High-school mathematics RCT in Turkey. Not a 5–10 trial.

[132]Erin A. Maloney et al., *Psychological Science* 26 (2015): 1480–1488. School homework-help study. Leave the counters in the child's hands.

this book, even if the birthday says otherwise. If the birthday says "third grade" and *altogether* still means add, stay. The types remain the classification. Geometry remains content. Early data remains a tally that tells how many.

Life of the habit — tips that are moves, not a ninth chapter.

Revisit next Tuesday. The same type, new cover story, smaller or larger numbers. A compare you did in October should still be a compare in March.

Mix last week's type with this week's, inside the math hour, every week you can. Mixing is how the child learns to choose. Blocked practice is for a new move. Mixed practice is for the rest of the life of the habit.

A path through elementary mathematics is the promise. A diploma is not. A percentile is not. Growing five-to-seven into eight-to-ten happens by skill, not by birthday. A ten-year-old may still be in the five-to-seven band of a chapter. A six-year-old may already name a compare and keep a skinny triangle. Place by what they can build.

Stop while they still have a sit left in them. Date the notebook. Keep one wrong answer you can reteach. The records chapter is that file. The resources chapter is fit, not rank. You do not need a franchise to keep Tuesday.

Five-to-seven talk and hands. Eight-to-ten a short written line as well. The question does not change: what kind of story is this?

Chapter 9

Records

The math *work* can be as serious as any kitchen's. The *credit* is a family claim until a stranger can use it — a receiving third-grade teacher, a Pennsylvania evaluator, a New York quarterly reader, a later high-school transcript the family will someday write. That stranger will not have sat at your table. They will have a page.

This chapter is how you make that page at ages 5–10. Lawful, honest, and readable to a stranger are three jobs. Keep them in view.

This week you can date one task in a notebook so a stranger could almost read it. Today the student tries the task. You write the line.

What you are making

You are making a **dated notebook of mathematics**, with the task named, plus an **exit ticket**, plus **one diagnostic item**, plus optional publisher placement to start a purchased program or restart after a gap. You are not making a national diploma. You are not making Little Thinkers Math I.

There is often **no** elementary transcript. When a stranger asks, the title is **Mathematics** or **Arithmetic**. Never a brand. Never Singapore 2A. Never Beast Academy 3. Never Math-U-See Beta. Never Saxon 5/4. Never RightStart C. Never Khan Kids. Never CTCMath Grade 3. Never **Little Thinkers Math I**. The dated notebook and the exit ticket live *inside* Mathematics or Arithmetic.

There is no federal homeschool diploma and no national mathematics credit at this age.[133] Texas, North Carolina, and New York do not issue one. Pennsylvania's supervisor or approved-organization diploma after listed graduation courses is a high-school object, not a five-year-old's file. Virginia's list of subjects is a list, not a transcript. A parent-issued diploma can be a real piece of paper later. It is not a registrar's national arithmetic credit now.

Two readers, even at 5–10, if anyone outside the kitchen asks. The **state**, in a thick jurisdiction: an IHIP, an affidavit, a portfolio, a test in reading and mathematics. A **later stranger**: a public school on transfer; a private school; a future high-school transcript. State compliance is not a math diploma. A thick Pennsylvania portfolio can be in statutory compliance with no franchise score. A silent Texas year can be lawful with no file at all. Neither fact is a reason to skip the moves. Both facts are a reason not to fake a credit.

A family that invents "Little Thinkers Math I, 1.0, A" for a seven-year-old has created a slogan. A receiving teacher cannot map it. Athletic eligibility and college exams are the wrong object for this book. They belong to an older year and to *Mathematics for Homeschooling*. A parent of a nine-year-old who is already anxious about core-course worksheets has been handed the wrong timeline.

The national survey of homeschool subjects offered an **arithmetic** checkbox. In 2016, arithmetic was taught that year to 83 percent of K–2-equivalent homeschoolers and 86 percent of the 3–5 band.[134] Informal learning does not arrive as "Mathematics, 1.0, B+." If a thick state later wants a log, the translation is: name the task, date it, keep a sample.

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[133] No national homeschool diploma: Texas Education Agency Home Schooling and Alternative Schooling pages (state does not award a homeschool diploma); North Carolina DNPE (the home school issues any later diploma; the State does not); Pennsylvania 24 P.S. § 13-1327.1(d.1) diploma is a high-school object; New York 8 NYCRR 100.10 (home instruction does not yield a local or Regents diploma); Virginia list of subjects is a list, not a transcript. Access date for URLs in these notes: 3 September 2026.

[134] Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey*, NCES 2020-001, Table 9: arithmetic taught that year, 83 percent in K–2, 86 percent in 3–5, 64 percent of all K–12-equivalent homeschoolers. Quote the arithmetic row. Computer science, history, inquiry, and foreign-language percents from the same table are other subjects. The 2023 First Look (NCES 2024-113) did not republish subject-taught tables.

Titles a stranger can read

Print these and only these as elementary names, if a name is needed at all:

Mathematics or **Arithmetic**. **Reading / Language Arts. Science. Social Studies / History** (if taught). **Art, Music, Physical Education.**

Those names travel. They are the names public elementary reports already use. They are the names homeschool elementary files, when they exist, already follow.

Never a brand. Never Little Thinkers Math I.

The brand, if used at all, sits in a parenthetical or in a description: "Mathematics (pasta number line; Primary Mathematics 2A as the material; kitchen measure)." Combining objects is allowed. Combining does not mint a new title. Kitchen number line plus library book plus one Math-U-See lesson is still Mathematics.

A publisher's "grade 2 math workbook" is a scope, not a legal grade, and not a transcript line. Placement is by skill, not birthday. Instruction for a six-year-old need not be "first grade."[135]

Here is an illustration, labelled as such, not a reported family:

> 3 September 2026. Pasta number line to 20. Task: 7 + 5. Child built 7, added 5, wrote 12. Missed 9 − 4; rebuilt with blocks; wrote 5.

That is a record a parent, a Pennsylvania evaluator, or a New York quarterly narrative can actually use. It is not a credit. It is not a sitting on a college inventory. It is not a diploma.

The law is a pattern, not one form

U.S. homeschooling is state law. There is no federal math office. Dates and subject lists change. Read your current department page and the statute it cites. A color-coded chart on a membership site is not your statute. Five official-page examples show the range. **None of them requires a critical-thinking course.** Math or

[135] New York State Education Department practice: instruction for a six-year-old need not be "first grade." CCSS-M: the standards do not dictate curriculum or teaching methods; what students can learn depends on what they have learned before. Map, not a homeschool statute. NMAP Finding 15: developmental appropriateness is contingent on prior opportunity to learn.

arithmetic is the usual required or assumed elementary object.[136]

Texas. The Agency does not regulate home-school programs. Parents follow a bona fide written curriculum: reading, spelling, grammar, **math**, and good citizenship. *Leeper*, restated on TEA's Alternative Schooling page, names math. Critical thinking is not named. The state does not award a homeschool diploma. At 5–10 there may be no external reader at all. The kitchen still benefits from a dated notebook if the family later transfers. Transfer is treated like an unaccredited private school. "We did Beast Academy 2" is not, by itself, a public-school credit. A district may test for placement.

North Carolina. Notice of Intent. Nine calendar months. Each year, a nationally standardized test in English grammar, reading, spelling, and **mathematics —** not a franchise, and not critical thinking. The home school, not the State, issues any later diploma. A five-hour day is a recommendation, not the statute, and not a math-minutes table. DNPE tells the family not to name the homeschool after a curriculum brand. A thinking notebook is extra. It is not the annual test. The annual battery is compliance. It is not this book's teaching check, not NAEP, and not a percentile promise.

Pennsylvania. Elementary **shall include arithmetic**, along with English, science, geography, history of the United States and Pennsylvania, civics, safety, health, physical education, music, and art. No critical thinking. No "Little Thinkers Math." Portfolio tests in grades 3, 5, and 8 are in reading/language arts and **mathematics**, not a franchise sitting. Drop samples into the log of materials. Do not invent a parallel "Little Thinkers" portfolio. 180 days or 900 hours elementary are whole-program hours, not a math-minutes table.[137] A supervisor diploma is Pennsylvania's, and it is high school.

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[136] Texas *Leeper*, restated on TEA Alternative Schooling: reading, spelling, grammar, **math**, good citizenship. North Carolina DNPE: annual nationally standardized test includes **mathematics**. Pennsylvania 24 P.S. § 13-1327.1: elementary shall include **arithmetic**; portfolio tests in grades 3, 5, and 8 in reading/language arts and mathematics. New York 8 NYCRR 100.10(e)(2)(i): grades 1–6 require **arithmetic**; a unit is 6,480 minutes; 900 hours grades 1–6; quarterly reports; annual assessment named tests or narrative; composite above the 33rd percentile or one year of growth. Virginia § 22.1-254.1: parent writes a list of subjects; evidence of progress by 1 August; children under six as of 30 September exempt from evidence-of-progress. None of the five requires a named critical-thinking course. This chapter is not legal advice. The parent reads their own statute.

[137] Hours of elementary math instruction in 2022–23 are not in the opened First Look subject/time tables. North Carolina's five-hour day is a recommendation across the day. Pennsylvania's 900 elementary hours are whole-program hours. There is no national math-minutes table in these pages.

Virginia. Notice by 15 August, a parent-written **list of subjects**, and evidence of progress by 1 August: a composite score at or above the fourth stanine, or an equivalent SAT, ACT, or PSAT score, or an evaluation or college transcript the superintendent accepts. Mathematics is not named as required because no subject is named as required; the parent writes the list. Mathematics is the ordinary word to write. Critical thinking is not named. Those sittings are not this book's short check. Children under six as of 30 September are exempt from the evidence-of-progress subsection. A publisher placement test does not, by itself, satisfy Virginia's fourth-stanine object unless the parent has chosen a nationally normed battery.

New York. Letter of intent, an Individualized Home Instruction Plan, quarterly reports, and an annual assessment. Grades 1–6 required subjects include **arithmetic**, reading, spelling, writing, English language, geography, U.S. history, science, health, music, visual arts, physical education. Arithmetic is named. Critical thinking is not named. Put the work under Arithmetic in the quarterlies: material covered in each IHIP subject. A unit is 6,480 minutes; a kitchen number line can be named under arithmetic. Home instruction does not yield a local or Regents diploma. Annual assessment: named commercially published tests or, in some grades, a written narrative. Composite above the 33rd percentile or one year of growth is a probation threshold in that regulation. It is not a goal this parent manual advertises as success.

If you live in none of those states, you still have a department page. Open it.

What a short check is

Three objects, not a battery.

1. **A dated notebook.** Date, task named, what the child built or wrote, one sentence of what happened. Under **Mathematics** or **Arithmetic** if a title is needed. A check that the child cannot yet read should be oral or built. A five-year-old's check is pasta and talk, not a silent twenty-item page.

2. **An exit ticket.** Two to four items at the end of a math hour, one of them not today's new skill. Right, or wrong-with-a-reason you can use tomorrow, is enough. Not a percentage identity.

3. **One diagnostic item.** A single item from a slightly earlier skill, or a slightly later skill, to see whether yesterday stuck or whether tomorrow is premature.

Not a forty-item diagnostic. Keep one wrong answer you can reteach. That is the file of the habit.

4. **Optional publisher placement** (the organization sells the test). Singapore untimed PDFs; Beast Academy unit tests; RightStart online or downloadable placement; Math-U-See yes/no tool; Saxon skill-level tests; Math Mammoth placement as the publisher describes it; CTCMath topic diagnostics. Use to start a purchased program, or to restart after a gap. IES does not certify them as randomised trials. They place by skill. They do not license a national percentile. They do not satisfy North Carolina unless the chosen test is a nationally standardized achievement test covering grammar, reading, spelling, *and* mathematics — a publisher math placement PDF generally is not that battery. They do not satisfy Virginia's fourth-stanine object by themselves.

A parent who feels rusty can turn the ticket into a performance with a tight jaw. A school homework-help study of first- and second-graders found that math-anxious parents who helped often were the ones whose children learned less math over the year; when help was infrequent, the association disappeared.[138] That is a school study, not a homeschool trial. The useful move is still: date the work, keep one wrong answer, leave the pencil in the child's hand while they try.

What a short check is not

NAEP. A percentile promise. A homemade high-stakes battery.[139] Terra Nova as the kitchen model. MAP as an identity. CLEP. AP. SAT. SAT Subject Tests — discontinued, last international sittings June 2021.[140] CCTST, Watson–Glaser,

[138]Erin A. Maloney, Gerardo Ramirez, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock, "Intergenerational Effects of Parents' Math Anxiety on Children's Math Achievement and Anxiety," *Psychological Science* 26 (2015): 1480–1488. First- and second-graders; school homework-help study, not a homeschool randomised trial.

[139]NCES 2024-113: 3.4 percent homeschooled in 2022–23, about 1,765,000; K–2 4.0 percent; 3–5 3.6 percent. Instruction-at-home 5.2 percent is a different bucket (homeschool or full-time virtual). NAEP 2024 Grade 4 mathematics: average 237; 39 percent at or above NAEP Proficient; 24 percent below Basic — NAEP, not a homeschool census. NCES has never published a nationally representative homeschool NAEP mathematics score.

[140]College Board, 19 January 2021 announcement: SAT Subject Tests discontinued; U.S. sittings ended immediately; last international administrations May and June 2021. No CLEP for elementary. No AP at this age. Those objects belong to *Mathematics for Homeschooling*.

Cornell Level X. A convenience-sample homeschool scoreboard. This book does not reprint that fight.

NCAA, Common App, and SAT math are the older book. Dual enrollment, Advanced Placement, and credit-by-exam are not this chapter. Scores help a stranger believe a *high-school* transcript. They do not replace a dated task at seven.

If they transfer

Transfer is a local placement problem. A brand in the title column will not be read as Mathematics. Another reason the title should already look like Mathematics, or Arithmetic.

A receiving teacher who sees "Little Thinkers Math I, A" has been given a slogan. A teacher who sees "Mathematics — dated notebook; pasta number line; 7 + 5 built and written; missed 9 − 4, rebuilt" has been given a year they can place.

Place by skill, not birthday

Can the child say how many without only reciting? If not, they are still in the counting chapter, whatever their age. Can they hear 12 for `8 + 4 = □ + 5` as an operational equal sign? If not, they are still in Chapter 3. Can they place 1/2 on a 0–1 line? If not, they are still in a sharing-to-number stage. High-sounding *titles* cannot skip those moves. A ten-year-old may still be in the five-to-seven band of a chapter. A six-year-old may already name a compare. Skill first. Publishers opened for this book say the same in their own placement language: beginning earlier is common and appropriate; levels rather than grades; Greek letters so the child is not "behind a grade"; an age table first.[141]

[141] Singapore Math Inc. placement page, accessed 3 September 2026: beginning at an earlier level is "common and perfectly appropriate." Art of Problem Solving / Beast Academy: age table first. Math-U-See: Greek letters so the child is not "behind a grade." RightStart: levels rather than grades. Placement tests are tools the organizations sell. IES / WWC does not certify them as randomised trials.

What to keep in the file

1. A dated notebook: date, task named, what the child did, one wrong answer you will reteach. Under **Mathematics** or **Arithmetic**.
2. Exit tickets from the week, not a crate of every worksheet.
3. One diagnostic item, labelled.
4. Optional publisher placement, if you used one to start or restart a program — as a placement, not as a yearly IQ.
5. State objects that actually apply: a North Carolina annual mathematics-inclusive battery if you are in North Carolina; a Pennsylvania grade 3, 5, or 8 mathematics portfolio sample if you are in Pennsylvania; a New York quarterly under Arithmetic if you are in New York; a Virginia list that includes Mathematics if you write the list. Compliance stays in its lane.

Life of the habit, as the file you keep, lives here: date the work; name the skill; keep one wrong answer you can reteach. Revisit next Tuesday is a move in Chapter 8. The notebook is how Tuesday leaves a trace.

A path through elementary mathematics is the promise. A diploma is not. A percentile is not.

Chapter 10

Resources

This chapter names programs families actually meet. It does not rank them. It does not sell them. It is a fit list so you can match an object to the table you already have: how much of the teaching you will carry, whether a kitchen can use it tonight, what it costs in money and in minutes, and whether the child will still build, sketch, and write.

A "grade 2 math workbook" is a publisher's scope, not a legal grade. Placement is by skill. The record, as Chapter 9 said, still says **Mathematics** or **Arithmetic**. Combining two honest objects does not mint a new title. Kitchen number line plus library book plus one lesson from a purchased program is still Mathematics.

Independent trials of these programs *in homeschool* remain scarce. Ratings from the What Works Clearinghouse, where they exist, are school-side. They are not a kitchen bake-off. Choose by fit: whether you can hear a wrong answer in that program's pictures, whether the student will generate the work, how much script you want to carry, budget, age label.

This week you can name *why* tonight's object fits this child — usually a number line of tape and a handful of pasta. Today the student tries that line. They are not on a shopping task.

How to read a program

Four questions do the work.

Parent load. Are you asking one question on pasta you already have? Sitting with a scripted lesson every day? Learning bar models well enough to hear a wrong bar? Booking a video as a second mouth?

Style. Number line and counters? Worktext the child can read? Blocks and a video? Comics and puzzles? Spiral mixed practice? Short videos for several siblings?

Cost shape. Nothing tonight. A book you already own. A print set whose dollars sat on a page this access day and will stale. Fees this book did not verify. Confirm a current price on the page before you buy.

Age label. A five-year-old is not a Beast Academy Level 1 default. A ten-year-old may still need a “grade 1” book if cardinality is the skill. Birthday is a weak prior.

Reach numbers on an organization’s page are that organization’s copy, not a homeschool census and not an effect size.

Kitchen table, nothing spent

A number line of tape on the floor or the table. Pasta, crackers, lids. A clock already on the wall. Generic coins counted at the table, stored after, because coins are a choking hazard for the youngest. A scrap of paper. That is tonight.

Fit. One child, low load, low cost. The store you were already visiting is the same shape: a price as a number, coins to make it, never a sermon about a food system. The walk you were already taking is the same shape: steps as a unit, a skinny triangle in a sign. The library you already use is the same shape: a counting book plus one question — how many are there, really? In 2016, 66 percent of homeschool households used the library as a source of curriculum and books (78 percent in K–2, 70 percent in 3–5). In 2022–23, 53 percent of the Homeschool-total row had visited a library in the past month, against 29 percent of all K–12.[142] Those are different questions in different years. Both are a free hour. NYPL and ALA language about libraries is mission, not a kit.

[142] NCES 2020-001, Table 6: library as a source of curriculum and books, 66 percent (K–2 78; 3–5 70). NCES 2024-113, Table A-5: Homeschool-total 53 percent visited a library in the past month versus 29 percent of all K–12. Do not swap 53 for 66. New York Public Library mission page; ALA Library Bill of Rights Article V (age). Mission, not a kit. Access date for URLs in these notes: 3 September 2026.

Transcript. Mathematics or Arithmetic. Never Little Thinkers Math I. Never "Consumer Math K." Never Outdoor Math 1.

The teaching chapters of this book are written for this object first. A purchased program is a fit, not a requirement.

Singapore Primary Mathematics and Dimensions Math

Concrete–pictorial–abstract, number bonds, bar models, mastery within a level. Parent load is moderate to high: bar models are a representation you must learn well enough to hear a wrong bar. Strip diagrams sit inside the Institute of Education Sciences grades 4–8 problem-solving guide, Recommendation 3, as a visual representation.[143] They are not a miracle.

The What Works Clearinghouse Singapore Math intervention report, December 2015, Primary Mathematics protocol, K–8: no studies of Singapore Math that fall within the scope of that protocol meet WWC group design standards. Because no studies meet WWC group design standards, the WWC is unable to draw any conclusions based on research about the effectiveness or ineffectiveness of Singapore Math on the achievement of primary students in kindergarten through grade 8.[144] Bar models are a representation. That sentence is not "Singapore works."

Editions of Primary Mathematics — U.S., Standards, 2022, Singapore International — are not interchangeable. The publisher separates their tests. Starting at an earlier level is, on the publisher's own page, common and perfectly appropriate. A ten-year-old in 1A is a placement result, not a scandal.

Dimensions Math is the same publisher family's comprehensive PK–8 series.

[143] John Woodward et al., *Improving Mathematical Problem Solving in Grades 4 Through 8*, NCEE 2012-4055 (IES, 2012; revised 2018), Recommendation 3 (strong): visual representations, including strip diagrams. On-age for 8–10. Bar models are this family of representation.

[144] What Works Clearinghouse, *Singapore Math®* intervention report, Primary Mathematics protocol, K–8, December 2015. Quoted finding: "No studies of Singapore Math® that fall within the scope of the Primary Mathematics review protocol meet WWC group design standards. Because no studies meet WWC group design standards at this time, the WWC is unable to draw any conclusions based on research about the effectiveness or ineffectiveness of Singapore Math® on the achievement of primary students in kindergarten through grade 8." Copy this finding whenever Singapore Math is named. Do not launder bar models into a curriculum endorsement.

Homeschool set prices sat on an opened page on 3 September 2026: Grade PK with Teacher's Guides $123.00; Grade K $137.50; Grades 1–5 with Teacher's Guides $152.00; Grades 1–5 with Home Instructor's Guides $169.00; Sprints workbooks $14.50.[145] Those are prices at access. They will stale. Confirm before you buy. Video subscriptions exist as a second mouth; that is the publisher's offering, not a trial.

Fit. A family that wants bar models and a mastery level, and is willing to learn the bars, or to buy the video as a second explanation. Not a statute. Not a clearinghouse endorsement. WWC Singapore Math, December 2015: no studies meet group design standards.

Transcript. Mathematics. Not "Singapore 2A." Not "Dimensions 3."

Beast Academy — age-labelled

Comic-guide plus practice, Levels 1–5, from Art of Problem Solving. Puzzle-hard. Parent load is moderate: the comics are readable; the puzzles are not a scripted parent lesson; some children will need the parent as reader. The publisher's age labels, copied not rounded: ages **6–8 Level 1**; **7–9 Level 2**; **8–10 Level 3**; **9–12 Level 4**; **10–13 Level 5**.[146] Levels tend to be slightly more difficult than grade level. If placing by grade, the publisher says to use the grade most recently completed. A five-year-old is below Level 1. Level 1 is not a five-year-old default. A student who completes Level 5, on the publisher's page, has the background for AoPS Prealgebra. That next step is older than this book's center. A seven-year-old is not sent to AoPS Prealgebra as the family hour. Age-label. This track is not a crown for gifted, and it is not the default pasta hour.

Fit. A family whose child thrives on hard puzzles, age 6 and up by the publisher's table, and who can hear a wrong answer in that style. 2026 public list price was not rendered on the placement page this access day; confirm on the current page.

Transcript. Mathematics. Not "Beast Academy 3B."

[145] singaporemath.com Dimensions Math PK–5 for homeschool catalog, accessed 3 September 2026. Set prices as listed in the body. Vendor prices at access; they will stale.

[146] Art of Problem Solving, Beast Academy placement and help pages, accessed 3 September 2026. Age bands: 6–8 Level 1; 7–9 Level 2; 8–10 Level 3; 9–12 Level 4; 10–13 Level 5. Levels tend to be slightly more difficult than grade level. Level 5 as background for AoPS Prealgebra is publisher language. List price not rendered on the placement page this access day.

RightStart Mathematics

Scripted lessons that guide the teacher day by day, an abacus grouped in fives and tens, card games. Parent load is high. Levels rather than grades, so a new student can enter by skill. Twenty-three additional materials, on the publisher's homeschool page. Placement is online or downloadable. A Finland study is mentioned on that page without a design, sample, or outcome extracted for this book; it is not copied here as an effect size.[147]

Fit. A family that wants games and an abacus and will sit daily with a script. High load. Not a statute.

Transcript. Mathematics. Not "RightStart C."

Math-U-See, Primer through Zeta

Build, Write, Say. Integer blocks, a video, a worktext. Parent load is moderate by design: the video and the blocks carry the first explanation; you still need to hear 1/2 + 1/3 = 2/5 as a size mistake. Primer, Alpha, Beta, Gamma, Delta, Epsilon, Zeta are the levels that sit in or next to ages 5–10. Epsilon is a fractions *level*, not "fifth grade." Greek letters so the child is not "behind a grade." Placement is a yes/no mastery tool. Advance when the topic is there.

Fit. A family that wants blocks and a script or video, and will still leave the blocks in the child's hands. Moderate load.

Transcript. Mathematics. Not "Beta."

Saxon Math, Primary K–3 and Intermediate 3–5

Incremental, spiral, mixed practice as the point. Parent load is real on the lesson and lighter than a method you must first learn; the mixed sets do choosing work this book wants. Skill-level rather than calendar grade. Homeschool split names — 5/4 and neighbors — belong later than this book's center; Primary K–3 and Intermediate 3–5 are the 5–10-adjacent objects.

[147] RightStart Mathematics homeschool and placement pages, accessed 3 September 2026. Levels rather than grades; AL Abacus; day-by-day lessons. Finland study named on the homeschool page; design, sample, and outcome not extracted; not copied as an effect size.

School-side, labelled school, not a homeschool trial: the What Works Clearinghouse Saxon Math report, May 2017, found **mixed effects**, five studies all *with reservations*, 8,855 students, grades **1–3 and 6–8**, average improvement index **+8** percentile points (range −**1 to +16**).[148] That is mixed, school, mostly not kindergarten, not home. Publisher ESSA language for Saxon K–5 is school-implementation evidence from the organization that sells the program, not a homeschool randomised trial.

Fit. A family that wants spiral mixed practice and can live with review that returns last week's skill. High load at first, then moderate.

Transcript. Mathematics. Not "Saxon 5/4."

Khan Academy Kids and Khan grade maps

Khan Academy Kids is described for ages 2–8, preschool through second grade. Khan grade-mapped courses are free practice and a map. Parent load is low to moderate. COPPA still applies; ages 5–10 are under 13; you hold the account. Khanmigo "never gives you the answer" is a design goal on an organization page, not a 5–10 trial.[149] A free map does not, by itself, produce Mathematics on a record unless you write the record: what was done, dated.

Fit. A family that wants free video and practice, and will still build with objects first. Low to moderate load.

Transcript. Mathematics. Not "Khan" as the credit.

[148] What Works Clearinghouse, *Saxon Math* intervention report, May 2017, Primary Mathematics protocol. Mixed effects; 5 studies all with reservations; 8,855 students; grades 1–3 and 6–8; improvement index average +8 (range −1 to +16). School-side. HMH ESSA "STRONG" language for Saxon K–5 is school-implementation evidence from the organization that sells the program, not a homeschool randomised trial.

[149] Khan Academy About and Khan Academy Kids pages, accessed 3 September 2026. Kids ages 2–8. Free content. Khanmigo "never gives you the answer" is publisher copy, not a 5–10 trial. COPPA: 15 U.S.C. 6501; general compliance date 22 April 2026, which has passed. Parent holds the account.

Math Mammoth Light Blue, and CTCMath

Math Mammoth Light Blue is indexed as a complete K–8 worktext curriculum the child can often read. A live fetch of the complete-curriculum page timed out on 3 September 2026; details here are from the search index, not a full opened HTML.[150] Parent load is low to moderate if the child can read the page. Confirm the current page before you buy. List price this access day is not claimed here.

CTCMath is a K–12 video-plus-interactive-questions system with reports and topic diagnostics. Parent load is low to moderate; several siblings can sit on one login. 2026 list price was not displayed on the opened homepage; confirm on the current page. The organization sells the program. Topic diagnostics are a tool, not an IES-certified trial.

Fit. Worktext for a reader; short videos for a household that wants one login. Neither is a crown.

Transcript. Mathematics. Not "Math Mammoth 3." Not "CTCMath Grade 3."

Other school programs you will see named

These are school-side What Works Clearinghouse ratings, labelled school, copied not rounded, not a reason to sell a franchise at the table.[151]

Everyday Mathematics — potentially positive effects, small extent of evidence, grades 3–5 in the meeting-standards study (Waite 2000).

DreamBox Learning — potentially positive, K–1, one study meeting standards.

Odyssey Math — potentially positive, grades 4–8.

[150]Math Mammoth Light Blue: search-indexed publisher description of a complete K–8 worktext; live fetch of the complete-curriculum HTML timed out 3 September 2026. CTCMath homepage, accessed 3 September 2026: K–12 video, interactive questions, reports; list price not displayed that day.

[151]WWC intervention reports, accessed 3 September 2026: Everyday Mathematics (2010/2015) potentially positive, small extent, Waite (2000) QED with reservations, grades 3–5; DreamBox Learning (December 2013) potentially positive, K–1, Wang and Woodworth 2011; Odyssey Math (January 2017) potentially positive, grades 4–8; Investigations in Number, Data, and Space (February 2013) potentially positive, grades 1–5, two studies, >8,000 students, improvement index average +2 (range −4 to +10). All school-side. Improvement-index exacts for Everyday Math 2015, Odyssey, and DreamBox are partly from report headers this pass.

Investigations in Number, Data, and Space — potentially positive, grades 1–5, two studies, average improvement index +2 (range −4 to +10).

None of these is a parent-at-home randomised trial. None is a kitchen winner. **Singapore Math**, named again so the sentence stays attached: WWC December 2015, no studies meeting group design standards; the WWC is unable to draw conclusions about effectiveness or ineffectiveness.

Extra help, age-labelled, as analogy not as a parent-as-TA

Named English maths trials, copied honestly, labelled by year group / age, as analogies for short, structured, frequent extra help — not as a parent teaching-assistant programme, and not as a ranking of kitchen curricula.

Ark Mathematics Mastery, Year 1 (~age 5–6): **+2 months**, 3 of 5 padlocks, on the Education Endowment Foundation project page (5,108 pupils / 90 schools in that figure).[152] Combined with a Year 7 trial, average +1 month.

1stClass@Number, Year 2 (~age 6–7): **+2 months** in two effectiveness trials, including pupils eligible for free school meals in the second trial.

Catch Up Numeracy: **no extra** compared with generic structured one-to-one. First trial: Catch Up g = +0.21 versus no extra 1:1; matched-time 1:1 g = +0.27; no significant difference between the branded kit and generic structured 1:1. Second trial: no evidence Catch Up beat other teaching-assistant 1:1; among pupils eligible for free school meals, Catch Up was 2 months *less* than other TA provision.

[152] Education Endowment Foundation project pages, accessed via memo 02 this research: Ark Mathematics Mastery primary (Jerrim, Vignoles, and Cowan 2015), Year 1, +2 months, 3/5 padlock; 5,108 pupils / 90 schools on the EEF project page (an 83 vs 90 school-headcount discrepancy is noted in the research brief; exact Hedges g not independently extracted). 1stClass@Number 1 (Edge Hill / Every Child Counts), Year 2, two effectiveness trials, +2 months each, including FSM in the second (NFER; 226 schools / 1,797 pupils in the second-trial write-up). Catch Up Numeracy (Rutt, Easton, and Stacey 2014 first trial): Years 2–6; Catch Up g = +0.21 vs no extra 1:1; matched-time 1:1 g = +0.27; no significant difference branded vs generic. Second trial: no evidence Catch Up beat other TA 1:1; FSM Catch Up 2 months less than other TA provision. Named *maths* trials only. Philosophy for Children attainment trials (including a 2021 effectiveness finding of 0 months on reading, with no evidence of maths impact) are not mathematics programs and are not copied here as math findings.

The portable claim: short, structured, frequent extra help can move numeracy; the branded kit is not always the active ingredient. A parent is not a teaching assistant with that training. These are school maths trials, age-labelled. Named maths trials only.

When you are the ceiling for this year's idea, a weekly tutor, a co-op arithmetic hour, or a video as a second mouth is ordinary. NCES 2016: any tutor 23 percent (19 percent in K–2 and in 3–5); any co-op 31 percent (25 percent in K–2, 32 percent in 3–5).[153] Unless the tutor's institution issues a transcript, you remain the person who dates the notebook. Name the tutor in a description if you want the help visible. The title is still Mathematics.

Placement tests are tools, not percentiles

Primary Mathematics untimed PDFs; Beast Academy unit tests; RightStart online or downloadable; Math-U-See yes/no; Saxon skill-level; Math Mammoth placement as described; CTCMath topic diagnostics. IES does not certify them as randomised trials. They place by skill. They do not license NAEP. They do not write Little Thinkers Math I. Use one to start a purchased program or to restart after a gap, as Chapter 9 said.

A child of ten may place into three different "grades" on three tests the same afternoon. That is three grains of "grade," not a contradiction. Start where the child can succeed, then add.

A short chooser, not a ranking

- If you want nothing spent tonight: tape, pasta, a clock, a library book. Title: Mathematics.
- If you want bar models and will learn them: Primary Mathematics or Dimensions. WWC Singapore Math, December 2015: no studies meet group design standards. Title: Mathematics.
- If you want puzzle comics, age 6 and up by the publisher's table: Beast Academy, age-labelled (6–8 Level 1, 7–9 Level 2, 8–10 Level 3, 9–12 Level

[153] NCES 2020-001, Table 3: tutor 23 percent (K–2 and 3–5 both 19); co-op 31 percent (K–2 25; 3–5 32); mother as main provider 78 percent (K–2 85; 3–5 78); formal curriculum 77 percent (K–2 81; 3–5 76). 2023 First Look did not republish these tables.

4, 10–13 Level 5). Title: Mathematics.

- If you want a script, an abacus, and daily sitting: RightStart. Title: Mathematics.
- If you want blocks, a video, and Build–Write–Say: Math-U-See Primer through Zeta. Title: Mathematics.
- If you want spiral mixed practice: Saxon Primary or Intermediate. School-side WWC mixed, +8 (−1 to +16), grades 1–3 and 6–8. Title: Mathematics.
- If you want free practice: Khan Kids (2–8) or Khan grade maps. You still write the record.
- If you want a worktext a reader can use: Math Mammoth Light Blue, indexed. If you want short videos and one login: CTCMath.

None of those bullets is "best." Fit is whether you can hear a wrong answer in that program's pictures, and whether the student will generate the work. When the student has unfinished cardinality, go back, whatever the calendar says. When you are the ceiling, buy a tutor, a co-op seat, or a video as a second mouth. When the destination is a stranger with a page in hand, title the year Mathematics or Arithmetic — the mathematics, not the franchise.

A path through elementary mathematics is the promise. A diploma is not. A percentile is not. A catalogue star is not a finding.[154]

[154] Cathy Duffy Reviews is a catalogue with disclosed review copies and affiliate links; Rainbow Resource is a retailer catalogue. Neither is a trial. No homeschool-curriculum randomised trial comparing Singapore, Beast Academy, RightStart, Math-U-See, Saxon, Khan, Math Mammoth, or CTCMath was opened for this book.

A Note on Sources

Studies named in the chapters are listed in Notes, in one series at the back. That is where the full citations live, so the teaching pages can stay a teaching voice.

Some items the research behind this book did not open stay out of the teaching voice. I did not invent a coefficient, a statute, or a product feature to fill a hole. If a study is in the chapter, it is in the notes. If we could not open it, it is not used as a finding here.

Program hours, credit claims, and product descriptions in the resources chapter are taken from the companies' own pages. Access date 3 September 2026.