Math for Emerging Adults

Michael Gannotti

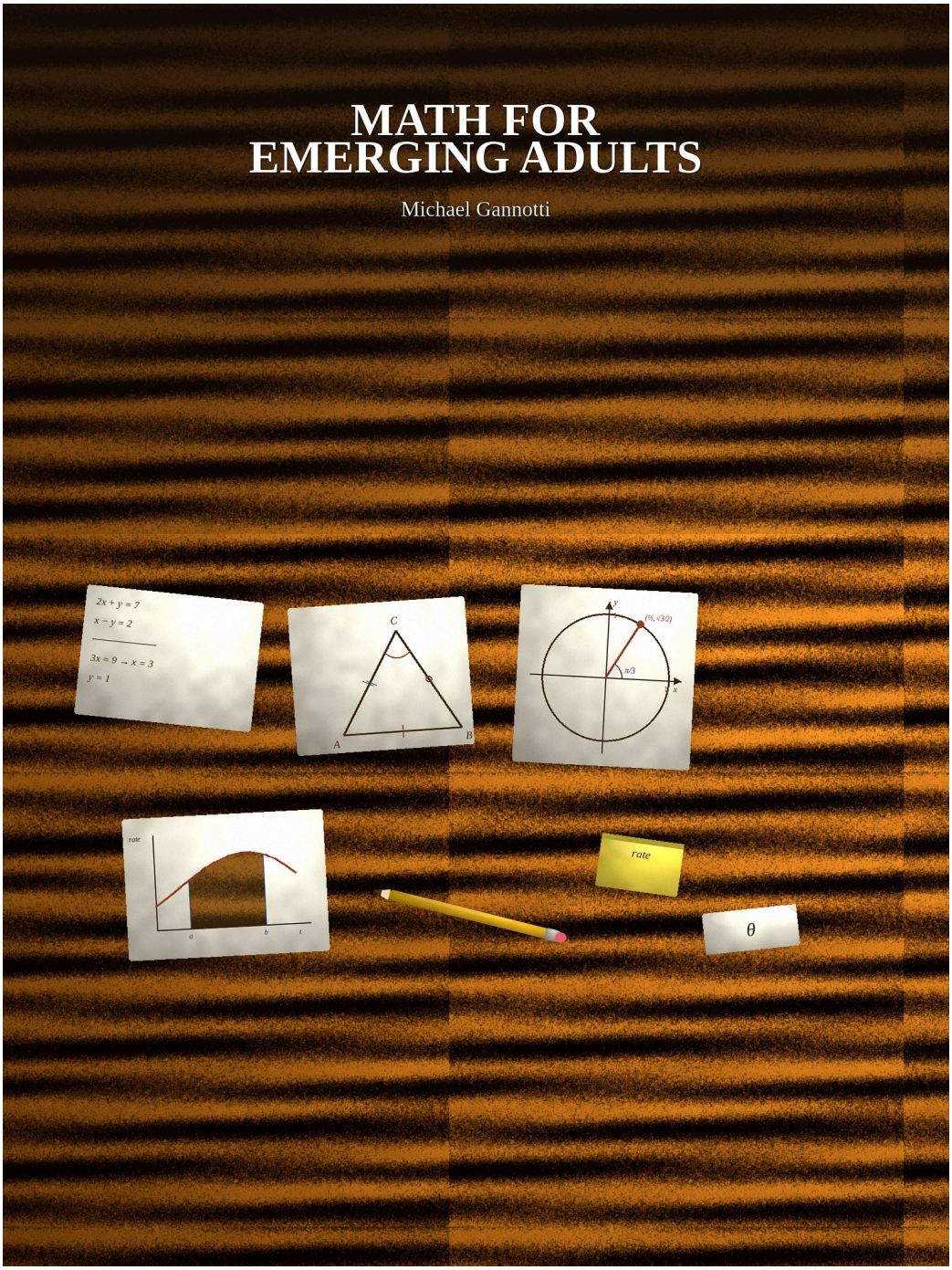

Figure 1: Cover

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This illustrated edition is the teaching-manual of 5 September 2026. Endnotes run in one series. Chapter-opening images are original still-lifes, not portraits of persons or children. Documents still constrain the facts.

Contents

Welcome

This book exists because the math hour at your table is the whole program. There is no math department down the hall and no specialist waiting after lunch. There is you, a fifteen- to eighteen-year-old, and today's idea — a solved equation with structure, a short proof with reasons, a function family read from a table, a unit-circle location, a rate that becomes instantaneous, or a sample number that is not yet a population claim. That is enough, if you know what to do with the hour.

I wrote this for a capable, busy, willing parent of a fifteen- to eighteen-year-old. You may be teaching two ages at once, fifteen-to-sixteen and seventeen-to-eighteen, at one table. You may be fitting a math sit between a job, a younger sibling, and a store run. You may love algebra, or you may remember high-school math as a fog of worksheets and a birthday that somehow meant Calculus. Many adults feel rusty when they sit down to teach high-school math they last saw as a student. That feeling is common. It is not a verdict. We will move on from it. This book will make you fluent enough in today's idea to hear a wrong turn, and to ask a good question — without taking the pencil.

You do not need to be a mathematician. You do need to hear letter-moving with no structure as a missing-meaning move, not a cute slip. You need to hear formula-sheet geometry with no reason, "move and flip" as magic, quadratic formula before a graph or table, a regression button offered as family ID, SOHCAHTOA as the whole of trig, power-rule week with no rate story, a sample mean offered as a population claim, and a birthday offered as Calculus readiness. You need a session shape you can run on a Tuesday. You need a few sentences that actually help. That is the job. A path through ordinary high-school challenges — Algebra I with linear and quadratic meaning; geometry with reasons; Algebra II function families; precalculus and trig with the unit circle; calculus with rate, accumulation, and the Fundamental Theorem for ready students; statistics and inference talk — is possi-

ble at a kitchen table. The path is not a personality trait and it is not a percentile. It is a small set of moves, practiced with you in the chair.

About 3.4 percent of U.S. students ages 5–17 were homeschooled in 2022–23 — roughly 1.765 million children.[1] In the high-school grades, ages that map to grades 9–12, the cell is about 3.1 percent.[2] That figure is context, not a ranking. Many readers of this book will not be in that pond. They will be after-school, weekend, or kitchen-table parents of enrolled adolescents. The moves still fit. Homeschooling is legal in all fifty states and the District of Columbia. The paperwork is not one load: Texas asks almost nothing of the state agency; New York asks for a written plan and regular reports.[3] There is no national homeschool diploma and no national "Emerging Adults Math I" credit.[4] Math is the usual required or assumed high-school object.[5] In the last federal tables that listed subjects taught at home,

[1]Ellena Sempeles and Jiashan Cui, *Parent and Family Involvement in Education: 2023*, NCES 2024-113 (Washington, DC: National Center for Education Statistics, September 2024), Table A-6: 3.4 percent homeschooled, approximately 1,765,000 students, ages 5–17 with a K–12 grade equivalent, 2022–23; grades 9–12 cell 3.1 percent. https://nces.ed.gov/pubs2024/2024113.pdf. Access date for URLs in these notes: 5 September 2026. Latest federal count this book uses; not a 2026 national total. A neighboring 5.2 percent received instruction at home (homeschooled or full-time virtual) and is a different bucket.

[2]Ellena Sempeles and Jiashan Cui, *Parent and Family Involvement in Education: 2023*, NCES 2024-113 (Washington, DC: National Center for Education Statistics, September 2024), Table A-6: 3.4 percent homeschooled, approximately 1,765,000 students, ages 5–17 with a K–12 grade equivalent, 2022–23; grades 9–12 cell 3.1 percent. https://nces.ed.gov/pubs2024/2024113.pdf. Access date for URLs in these notes: 5 September 2026. Latest federal count this book uses; not a 2026 national total. A neighboring 5.2 percent received instruction at home (homeschooled or full-time virtual) and is a different bucket.

[3]Texas Education Agency, "Home Schooling," https://tea.texas.gov/families-and-students/finding-school-your-child/home-schooling. New York State Education Department, "Home Instruction Questions and Answers," https://www.nysed.gov/nonpublic-schools/home-instruction-questions-and-answers. Homeschooling is legal in all fifty states and D.C.; these two pages illustrate a light-process and a heavier-process pattern, not a national template. This book is not legal advice.

[4]Texas Education Agency, "Home Schooling": the State of Texas does not award a diploma to students that are home schooled. New York State Education Department, "Home Instruction Questions and Answers": a high school diploma may only be awarded to a student enrolled in a registered secondary school. North Carolina Division of Non-Public Education, "Home School Records Retention & Diplomas FAQs," https://www.doa.nc.gov/divisions/non-public-education/home-schools/faqs/records-retention-diplomas: the State does not issue a diploma or a transcript; the home-school chief administrator does. "Emerging Adults Math I" is not a credit in any opened statute.

[5]Texas Education Agency, "Alternative Schooling," restating *Leeper*: bona fide instruction us-

the high-school math rows for grades 9–12 that year were Algebra I at 33 percent, Algebra II at 33 percent, Geometry at 26 percent, Calculus at 13 percent — and arithmetic still at 31 percent.[6] The hour in front of you is the work. Placement into Algebra I, Geometry, Algebra II, Precalculus, Calculus, or Statistics is by skill, not by birthday.

This book is not a reprint. *Math for Young Minds* is for ages 11–14. One pointer, if you have a younger sibling at the table. Then we teach at fifteen to eighteen. *Math for Little Thinkers* is for ages 5–10. One pointer, if you need the early years. Then we teach here. *Mathematics for Homeschooling* covers grades 1–12. It compressed high school into survey chapters. One pointer, if you need the whole-span map. Then we teach here. This book is not a sequel. *Critical Thinking for Emerging Adults* is the same house, a different subject and a different age band — Arnett's 18–25 is not this band. One pointer, then we teach this age's math.

Two refusals belong on the first day. Birthday is not Calculus readiness. An eighteen-year-old missing linear and quadratic structure does Algebra I repair without apology; a fifteen-year-old ready for limits may begin calculus meaning with supervision — skill and motivation, not a race.[7] Letter-moving and formula-sheet geometry are not high-school mathematics with meaning. Structure, reasons, and

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ing a written curriculum that includes math. North Carolina Division of Non-Public Education: annual nationally standardized test including mathematics. Pennsylvania 24 P.S. § 13-1327.1: elementary and secondary subject lists include arithmetic or mathematics. Virginia Department of Education, "Home Instruction": the parent writes a list of subjects. New York 8 NYCRR 100.10: required subjects by grade band include arithmetic or mathematics. None of these five opened maps names a critical-thinking course. Math is the usual required or assumed high-school object. The parent reads their own statute.

[6]Jiashan Cui and Rachel Hanson, *Homeschooling in the United States: Results from the 2012 and 2016 Parent and Family Involvement Survey*, NCES 2020-001 (Washington, DC: National Center for Education Statistics, 2019), Table 9: among grades 9–12 homeschoolers that year, Algebra I taught 33 percent; Algebra II taught 33 percent; Geometry taught 26 percent; Calculus taught 13 percent; arithmetic still taught 31 percent. Quote the high-school math rows. Do not reuse the elementary arithmetic 83/86 cells or the middle 66/41 cells as this band. The 2023 First Look did not republish the subject-taught tables. https://nces.ed.gov/pubs2020/2020001.pdf.

[7]National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Critical Foundations of Algebra: proficiency with whole numbers, fractions (including decimals and percents), and certain aspects of geometry and measurement — continuity if unfinished. Finding 10: conceptual understanding, computational fluency, and problem-solving skills belong together. Finding 15: claims that children of particular ages cannot learn certain content because they are "too young," "not in the appropriate stage," or "not ready" have consistently been shown to be wrong if prerequisite knowledge is present. https://files.eric.ed.gov/fulltext/ED500695.pdf.

representations come before the shortcut.

What a good math hour looks like

You sit down already knowing today's idea. An object is on the table — a solved problem on scrap paper, a marked geometry diagram, function-family table cards, a unit circle, a rate or accumulation sketch, sample-versus-population sticky notes. The student warms up on facts they can already get right. You model one solved problem out loud, short — graph or table visible beside the algebra. Then they try, and you wait. You ask one good question — not "did you get it?" but "If this is linear, what stays constant — the rate or the second difference?" or "What are we given? What must we show? What bridge gets us there?" Three seconds feels long. It is the work. Practice mixes the new move with last week's. The hour ends with one or two items they do alone. You stop talking sooner than feels polite. The student talks and builds. You hold the key. That is the hour. Forty-five to seventy-five minutes is the usual ambition; ninety when a proof or a calculus investigation needs it. The next long piece of this front matter, *The Math Hour*, will teach it in full. Later chapters will not reinvent it.

Figure 2: A kitchen-table math hour with a solved problem and a marked diagram

Students this age already do high-school mathematics. They do not become Calculus-ready because a candle was lit, and they do not become algebra-fluent because they can chant “move and flip.”

What you will actually get

Each teaching chapter does eight jobs, always in the same order, so you are never hunting for the move.

You will learn why this week’s idea is worth the struggle. You will understand it yourself, with the wrong answers you should be able to hear. You will get a session you can run this week: exact wording, named try-its in and out of the home, and a talk box. Your student will get a short section of their own. If it isn’t clicking, three diagnostics and a next move. Tools, including AI, stay optional and adult-

side. And you will get a plain checklist for “done enough,” so you can place by skill rather than by birthday.

The six teaching chapters follow the work: Algebra I; geometry with reasons; Algebra II function families; precalculus and trigonometry; calculus with meaning for ready students; statistics and inference talk. Records and resources come last. Calculus is for the ready — not every eighteenth birthday.

This week you can learn the session shape well enough to hear letter-moving with no structure. Today the student can solve a linear or friendly quadratic narrating each legal move, or sort a table as linear versus quadratic by differences.

What this book will not do

This book will not hand you 180 days of worksheets. Coverage is a map. Depth is the hour.

It will not sell you a curriculum. Later, a short resources chapter names common programs by fit so you can choose. Combining does not turn a brand into a course name. The transcript still says Algebra I, Geometry, Algebra II, Precalculus, Calculus, or Statistics — only if the year’s work was that course.

It will not treat a birthday as Calculus readiness, letter-moving as algebra with meaning, a formula card as geometry, a regression button as family ID, or an AP score as the only path that counts.

It will not promise a score. There is no guaranteed percentile, no math diploma, and no certificate of fluency inside these pages. What this book can promise is a path: the ideas in order, at the skill the student actually has, until they can do the next one unaided.

It is not “Emerging Adults Math I.” One math-hour AI box lives in the next long piece. Later chapters point back.

It is not a culture-war pamphlet. Safe math objects only: a solved equation, a marked diagram, a function-family table, a unit circle, a rate or accumulation sketch, a sample-versus-population sticky, a car-loan row, a bridge angle, a sports rate, a paycheck percent. Live fights stay off this table.

And this is not a book that lectures your student about you. The student is a person, not a percentile.

The promise

If you only remember one sentence, remember this: a path through high-school mathematics is the promise. A diploma is not. A percentile is not. An AP score is not the only path that counts.

The work is to take this student through mathematics they can actually use — where a quadratic lives on a graph and a table, what a reason is in a proof, how you know a family without a regression button, where an angle sits on the unit circle, what an average rate is becoming and in what units, whether a number is about the sample or a claim about the population — without skipping the gate because they are "ready for Calculus," and without parking them in a workbook that is too easy because a catalog printed a grade on the cover. You will sometimes slow down. You will sometimes skip ahead. Both are teaching. Birthday is not placement. Calculus is by readiness.

Three things have to hold. You understand today's idea well enough to hear a wrong turn. The student attempts first, with a representation, then the written equation or reason. Any helper — including an AI tool — stays on your side of the table: a supplement you host and constrain, not a partner during the attempt, not a photo-to-key, and not a secret friend.

You can do this. You do not have to know next year's idea today. You have to know this week's idea well enough to sit still while they struggle, then ask one good question. Start here. Read *How to Use This Book This Week*, then the one-page list of five things, then *The Math Hour*. After that, open the chapter that matches the skill in front of you. You will know more after one chapter than you know this morning. Your student will have something to try today. We can do this.

How to Use This Book This Week

Start at the skill in front of you, not on page one because a catalog, a birthday, or a well-meant relative said so.

This book is a handbook you open to this week's idea, not a novel you read cover to cover. The six teaching chapters follow the work, not twelve thin grade labels. Records and resources come last. Age bands here are 15–16 and 17–18. They tell you the grain of the same ideas. They do not tell you where *this* student sits.

This book does not replace the program already on the shelf. If a math curriculum, a kitchen solved-problem habit, a library placement pretest, or a dual-enrollment seat you already use is working, keep it. Use this book to hear a wrong turn, to run the hour, and to name the work a stranger can read. Combine honestly. The title is still Algebra I, Geometry, Algebra II, Precalculus, Calculus, or Statistics — only if the year's work was that course. Never a brand. Never Emerging Adults Math I.

Two ages at one table

You may be teaching a fifteen-to-sixteen-year-old and a seventeen-to-eighteen-year-old in the same sit. That is ordinary. The session shape in *The Math Hour* still holds; the object on the table and the grain of the question change.

The younger student talks and builds. A linear or friendly quadratic with justification, a short proof with reasons in a chosen format, a table sorted into a family without a regression button, right-triangle trig or unit-circle edges, sample-versus-population talk — that is enough. The older student can hold a little more: completed Algebra II and Precalculus function and trig work, and — if ready by skill — limits, rate, accumulation, and the Fundamental Theorem, with stronger infer-

ence talk. You may run a fifty-minute Algebra I repair sit with one student and a seventy-minute unit-circle sit with the other. You do not need two personalities. You need two first problems.

Pick the chapter by skill, not by birthday

Open the chapter you think is right. Skip to **What "done enough" looks like** at the end of the *previous* chapter, or to **If it isn't clicking** in the one you opened. If the student can already do those checks unaided, you are too early. If the checks from two chapters back are still failing, drop back.

If they still treat the equal sign as "write the answer" and cannot graph a friendly quadratic from a table, they are still in the Algebra I work of the first teaching chapter, whatever their age. Students this age already do mathematics. What they can do depends on what they have already been invited to try, not on a birthday.[8]

If they cannot yet write a short proof with reasons, they are still in the reason stage of geometry, whatever the birthday. An eighteen-year-old missing linear and quadratic structure does Algebra I repair without apology. Place by skill, not birthday. Critical Foundations continuity still applies if unfinished.

If they can already solve linear and quadratic equations with justification, write a short proof with reasons, and sort function families from tables, and they are still in a "grade 11 Algebra II workbook" only because the cover says so, skip ahead. A publisher's grade label is a scope, not a legal grade. Common Core State Standards for Mathematics is a map of typical U.S. public-school placement, not a homeschool statute.[9] The College Board AP Calculus Course and Exam Description is a College Board map, labelled — not a kitchen trial and not a birthday gift.[10]

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[8]National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Finding 15: claims that children of particular ages cannot learn certain content because they are "too young," "not in the appropriate stage," or "not ready" have consistently been shown to be wrong if prerequisite knowledge is present. Access date for URLs in these notes: 5 September 2026. https://files.eric.ed.gov/fulltext/ED500695.pdf.

[9]Common Core State Standards for Mathematics (Washington, DC: National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010): "These Standards do not dictate curriculum or teaching methods"; what students can learn at a grade "depends upon what they have learned before." A map of typical U.S. public-school placement, not a homeschool statute. http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf.

[10]College Board, *AP Calculus AB and BC Course and Exam Description* (effective Fall

Use this book's checklists, then teach. Calculus is for ready students. This book does not dump it as an eighteenth-birthday present.

Publishers say the same in their own placement language (AoPS readiness checks; Teaching Textbooks trial; CTCMath diagnostic). Those are placement aids, not a yearly identity.

How you use the parent half

Most of each teaching chapter is for you. Read it *before* the lesson, not over the student's shoulder.

Why this matters tells you what this idea unlocks. **For the parent: understand it yourself** gives one everyday picture, one precise picture, and three to five wrong answers you should be able to hear. Sit with those.

How to teach it this week assumes the session shape from *The Math Hour*. It will not rebuild the hour. It will give you this week's named try-its — at least one in the home and one outside it — the wording for *this* idea, and a talk box: an exact opening question, three to five follow-ups, how to wait, and what a stuck silence usually means. Age-band moves for 15–16 and 17–18 live inside that heading.

Practice that actually builds learning names the try-its with time, materials, safety, the fun, and the skill. Cars, sports, maps, and making *are* the practice. They still have to land on the equation, the proof, or rate meaning. **Tools, including AI** is optional, short, and for the adult. The rules live once in *The Math Hour*. **What "done enough" looks like** is how you leave: unaided work, not a perfect Tuesday. Calculus done-enough is for ready students — not every eighteenth birthday.

You do not have to read the whole chapter tonight. You do have to read the parent half of *this week's* idea before you sit down.

"Lesson plans" means this week's session shape. "Assessment" means a short skill check, an exit ticket, a done-enough checklist, and optional publisher placement — not a 180-day dump, not a percentile battery, not an AP score as the only thing that counts.

2020): a College Board map of topics and practices for AP Calculus courses — labelled vendor map in this book, not a kitchen randomized trial and not a birthday requirement. https://apcentral.collegeboard.org/courses/ap-calculus-ab/course.

The five-minute parent warm-up

Five minutes. Student not yet in the chair. Phone face down.

1. Read today's idea until you can say it in one sentence.
2. Look at one representation yourself — the solved problem, the marked diagram, the function table, the unit circle, the rate sketch. Ask, out loud, what would count as a check. Stay off any tool until you have looked.
3. Glance at the "wrong answers you should be able to hear." Name the one you would have given at seventeen.
4. Write one sentence you will actually say. Not a speech. Example: "If this is linear, what stays constant — the rate or the second difference?" Or: "What are we given? What must we show? What bridge gets us there?"
5. Close the book to the student page. You are ready.

If you are learning the idea *while* they are stuck, you will talk too much. Prepare first. Then sit still.

How the student uses "For the student"

Every teaching chapter includes a short section written to the student, not about them. One or two pages. Warmer. Direct. What the idea is, a tiny worked example, two tries, an "explain it back" prompt, and one challenge. Hand it over after your short model, not instead of it. You stay in the room.

The student page is not something to send off with an unsupervised chatbot. The attempt is still theirs. You still hold the key.

In the home and outside it

Every teaching chapter names try-its at the table and try-its on a walk, at a bridge or building, on a sports sideline, with a map, or with a paycheck percent. Cars, sports, maps, and making motivate. They do not replace the equation, the proof, or rate meaning. No live controversy in the examples. A car-loan row is a math object. Sports stats are rates. Pharmacy dosage as rate is a composite illustration — not medical advice.

When to skip ahead

Skip ahead when this chapter's "done enough" checklist is already true *unaided*. Slow down when the same wrong turn repeats after a clear look. **If it isn't clicking** will give you three likely causes and a next move. Hearing a wrong answer, then asking a better question, then naming a better move if needed, is teaching.

For this week: pick the chapter by skill. Do the five-minute warm-up. Run the hour as *The Math Hour* describes it. Let the student page be theirs. Stop talking sooner than you want to.

If You Only Remember Five Things

Keep this page. The chapters will add wording, representations, and this week's questions. They will not replace these.

1. Birthday is not Calculus readiness and not Algebra II finished. Place by skill. Critical Foundations continuity if unfinished. A candle does not make someone Calculus-ready. An eighteen-year-old missing linear and quadratic structure does Algebra I repair without apology. A fifteen-year-old ready for limits may begin calculus meaning with supervision — skill and motivation, not a race.[11]

2. Understanding, fluency, and problem solving belong together. Algebra: solved problems, structure, strategies. Same to both sides — still live. Geometry needs reasons; proof formats are options (two-column, paragraph, flow), not dogma. A program that advertises only one strand is selling a fragment.[12]

[11] National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Critical Foundations of Algebra: proficiency with fractions (including decimals, percents, and negative fractions) is foundational for algebra — continuity if unfinished. Finding 15: claims that children of particular ages cannot learn certain content because they are "too young," "not in the appropriate stage," or "not ready" have consistently been shown to be wrong if prerequisite knowledge is present. Finding 4 and related discussion: progress through algebra topics depends on prior knowledge, not age alone. Access date for URLs in these notes: 5 September 2026. https://files.eric.ed.gov/fulltext/ED500695.pdf.

[12] National Mathematics Advisory Panel, *Foundations for Success*, Finding 10: conceptual understanding, computational fluency, and problem-solving skills belong together. Jon R. Star et al., *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students*, NCEE 2015-4010 (Washington, DC: Institute of Education Sciences, 2015), Recommendations 1–3: solved problems; structure; strategy choice — grades 6–12, on-age for this book. Michelle Cirillo and Patricio G. Herbst, "Moving Toward More Authentic Proof Practices in Geometry," *The Mathematics Educator* 21, no. 2 (2012): 11–33: focus on logical arguments; formats as means, not ends. NCTM, *Catalyzing Change in High School Mathematics* (Reston, VA: National Council of Teachers of Mathematics, 2018), GARP: proof communicates in a variety of ways.

3. The student talks and builds on a representation. You ask one good question and wait. Three seconds is a convention, not a sacrament.[13] After they stop, wait again. If the silence is hard, look at the graph, the diagram, or the table, not at the student. Struggle before rescue: ask, wait, hint, then model. You do not grab the pencil. You do not finish the item. A helper, including an AI tool, stays on your side of the table. It does not sit in the chair during the attempt.

4. Reasons in geometry; meaning in calculus. Geometry: demand a reason before you crown a format. Calculus for the ready: limits → rate → accumulation → Fundamental Theorem — not power-rule week with no rate story. Statistics: sample versus population; ordinary stakes. Hearing a wrong answer, then asking a better question, then naming a better move if needed, is teaching. Silence in the face of a keyword error is not kindness.[14]

5. A path through high-school mathematics is the promise. A diploma is not. A percentile is not. An AP score is not the only path that counts. If a stranger asks what you taught, the title is Algebra I, Geometry, Algebra II, Precalculus, Calculus, or Statistics — only if the year's work was that course. Never a brand. Never Emerging Adults Math I.[15] Everyday stakes. No live controversy on the table.

Cars, sports, and maps motivate. They do not replace the equation, the proof, or rate meaning. Calculus by readiness, not birthday.

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[13]Robert J. Stahl, "Using 'Think-Time' and 'Wait-Time' Skillfully in the Classroom," ERIC Digest ED370885 (1994), https://files.eric.ed.gov/fulltext/ED370885.pdf: think-time; about three seconds as a convention; classroom-origin. Mary Budd Rowe, "Wait-Time and Rewards as Instructional Variables," NARST paper, ERIC ED061103 (1972), https://files.eric.ed.gov/fulltext/ED061103.pdf, is elementary science class, not a homeschool trial and not a math randomized trial.

[14]Patrick W. Thompson, "Images of Rate and Operational Understanding of the Fundamental Theorem of Calculus," *Educational Studies in Mathematics* 26, no. 2–3 (1994): 229–74; Thompson, Byerley, and Hatfield on coherence of meanings — rate and accumulation before rule flashcards. College Board, *AP Calculus AB and BC Course and Exam Description* (effective Fall 2020): labelled College Board map (limits, derivatives, integrals, FTC), not a kitchen RCT. Sample-versus-population talk is the statistics object for every reader in this book; full AP Statistics depth is optional.

[15]Ellena Sempeles and Jiashan Cui, *Parent and Family Involvement in Education: 2023*, NCES 2024-113 (Washington, DC: National Center for Education Statistics, September 2024): no national homeschool diploma. Transcript titles in this book are Algebra I, Geometry, Algebra II, Precalculus, Calculus, or Statistics by content — never a brand, never "Emerging Adults Math I." An AP score is one optional path among others, not the only path that counts. https://nces.ed.gov/pubs2024/2024113.pdf.

The student attempts first. You hold the key. If this week needs a compass, this is it. Five things. Then sit down and teach.

The Math Hour

The hour has a shape. Learn it once. Later chapters will give you today's idea, today's try-it, and today's questions. They will not rebuild this hour. When a chapter says "run the session," it means this.

You do not need a school bell. You need a warm-up, a short model, a real attempt, one good question then wait, mixed practice, and an unaided exit. A fifteen-to-sixteen-year-old may finish nearer forty-five minutes; a seventeen-to-eighteen-year-old may need seventy-five, or ninety when a proof or a calculus investigation needs the room. The shape does not change.

Sit down having already done the five-minute parent warm-up from *How to Use This Book This Week*. You know today's idea well enough to hear letter-moving with no structure as a missing-meaning move. The student talks and builds. You hold the key. Forty-five to seventy-five minutes, most weekdays, plus one named try-it outside the home, is the right ambition.[16] There is no national table of home-school math minutes.

Life of the habit, as a session rhythm, lives here: revisit yesterday; mix last week; stop.

1. Warm-up

Five to ten minutes. Known facts, or a quick skill: factor a friendly trinomial; evaluate a function from a table; name the sine of a special angle on the unit circle.

[16] Session length in this chapter is pedagogical design for the kitchen table, not a national math-minutes study (none was found). North Carolina DNPE recommends (not law) five hours across all subjects; Pennsylvania statute names 180 days or whole-program hours. None of these is a national math block. Access date for URLs in these notes: 5 September 2026.

Already-right material. This is retrieval, not a test of character.

Say: “We’re going to start with things you already know.”

Ages 15–16: a quick linear check, a friendly factor, or a one-step same-to-both-sides repair. “What did we do to both sides? Take your time.” Ages 17–18: a quick unit-circle value, a parent-function transform named aloud, or a short average-rate from a table. “Where is $\pi/3$ on the circle — and what are the coordinates?”

If they already get these right untimed, you may use a brief timer — one to five minutes, not the lesson.[17] Say: “We’ll time this only because you already get these right. Ready?” If they do not already get them right, skip the timer. Fluency is accurate first, then sometimes timed. Never time the introduction of derivatives. A timed algebra or calculus kit is not fluency.

Keep this short. The warm-up is not the lesson.

2. Short model

Eight to twelve minutes. One solved problem, think-aloud, graph or table visible before or beside the algebra — Star’s first recommendation for algebra knowledge.[18] Then you stop. Show the move. Then fade.

Say: “I’m going to show this one short. Then you’ll try.” “Watch. Both sides stay the same. That’s why I can subtract 4 from both sides.” Or: “Look at the table

[17] Lynn S. Fuchs et al., *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*, WWC 2021006 (Washington, DC: Institute of Education Sciences, 2021), Recommendation 6, strong evidence: timed activities last 1 to 5 minutes; they are not the entire focus. Population is intervention-oriented; not a parent-hour trial for derivatives. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/WWC2021006-Math-PG.pdf. National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (Washington, DC: U.S. Department of Education, 2008), Finding 11: automatic recall frees working memory for more complex problem solving.

[18] Jon R. Star et al., *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students*, NCEE 2015-4010 (Washington, DC: Institute of Education Sciences, 2015), Recommendation 1: use solved problems to engage students in analyzing algebraic reasoning and strategies. Recommendations 2–3: structure; strategy choice. Grades 6–12 — on-age for this book. National Mathematics Advisory Panel, *Foundations for Success*, Finding 27: explicit instruction has consistently positive effects for students with mathematical difficulties; this does not mean all instruction should be explicit. Finding 23: high-quality research does not support instruction that is entirely student-centered or entirely teacher-directed. https://files.eric.ed.gov/fulltext/ED500695.pdf.

first. Second differences are roughly constant — that's the quadratic family before we touch the formula."

On the second pass, leave a hole. When the idea is new, show one incorrect example — letter-moving with no structure, formula without reason, keyword grab, regression-button family ID, SOHCAHTOA-only trig, power-rule-first with no rate story, sample offered as population — and ask what went wrong. If you are still talking at minute thirteen, close the model. This is not a thirty-minute lecture.

3. Student attempt

This is the center of the hour. Fifteen to thirty minutes. One longer problem or a short set of *today's type*. Representation first, then the written equation or reason, as needed. Hand them "For the student" or the first try-it. Then you talk less than you want to.

Say: "This one is yours. I'll be quiet." Then be quiet. If they stall, use this order: ask, wait, hint, then model. Not the reverse. Ask: "If this is linear, what stays constant — the rate or the second difference?" Wait. Count a slow three. The silence is the work. If you fill it, you took the problem back.

Hint, one hint: "You already know factored form shows the roots. Try naming what this form makes visible before you solve." Then, if they are still stuck after a real try: "I'm going to show you this one step. Then you take it from here." Keep your hands off their pencil. The student talks and builds. You hold the key.

4. One good question, then wait

Two to four minutes of clock time that feels longer. The talk around it can run using the talk-box shape below. Not "did you get it?"

An authentic question is one for which you have not already written the answer. "What is the vocabulary word?" is recitation. Say: "What are we given? What must we show? What bridge gets us there?" Or: "Is this table linear, quadratic, or exponential — how do you know without a regression button?" Or: "As the interval shrinks, what is this average rate becoming — and in what units?"

Then wait. A slow three after you ask. A slow three again after they stop. Robert Stahl called that silence *think-time*; that label is classroom-origin, not a home-

school trial. About three seconds is a convention, not a sacrament.[19] Mary Budd Rowe found the same pause in elementary science class. That was science class. This book uses the pause so already-present reasoning can be heard.[20] If they are mid-reason, do not cut them off. If the silence is hard, look at the graph, the diagram, or the table, not at the student. One question. Maybe a follow-up. Then stop.

5. Mixed practice

Ten to fifteen minutes. Yesterday and last week mixed with today. Not forty of the new item.[21]

Say: “Two from today. One from last week. Then we stop.”

Cars, sports, maps, and making can sit inside this mix. They do not replace the equation, the proof, or rate meaning.

6. Exit ticket

Three to five minutes. Two to four items, one of them yesterday’s skill; one “explain in a sentence.” Book closed. No hints. No chatbot. Say: “These few are yours. I won’t help. That’s the point.” Done-enough is “right, or wrong-with-a-reason we can use tomorrow,” not a percentage.

Praise a clear structure talk, a reason stated before a format, a family named from differences or ratios, or a mind changed after a check — not speed. Hands in your lap. If a wrong answer keeps coming, hear it, ask a better question, then name a

[19] Robert J. Stahl, “Using ‘Think-Time’ and ‘Wait-Time’ Skillfully in the Classroom,” ERIC Digest ED370885 (1994), https://files.eric.ed.gov/fulltext/ED370885.pdf: about three seconds as a convention. Classroom-origin. Not a homeschool randomized trial.

[20] Mary Budd Rowe, “Wait-Time and Rewards as Instructional Variables,” NARST paper, ERIC ED061103 (1972), https://files.eric.ed.gov/fulltext/ED061103.pdf: elementary science classes. Rowe is science class. This book does not cite Rowe 1972 as a homeschool randomized trial or as a math trial.

[21] Fuchs et al., Recommendation 1: mix previously and newly learned material. Star et al., Recommendations 1–3. Woodward et al., *Improving Mathematical Problem Solving in Grades 4 Through 8*, NCEE 2012-4055 (Washington, DC: Institute of Education Sciences, 2012), Recommendations 2–3: visual representations; schemas, not keyword matching — continuity grain.

better move if needed. Hearing a wrong answer and leaving it uncorrected is not kindness.

If the hour goes sideways

Use this as a debug, not a verdict.

- Letter-moves with no structure → solved problem plus graph or table first; ask what the form makes visible.
- Keyword-grabs "of" or "altogether" → name the schema; draw the quantities.
- Treats "=" as "write the answer" → same-to-both-sides; solved balanced equations out loud.
- Formula-only geometry → "What's the reason?" before format; given / goal / bridge first.
- Regression-button family ID → sort tables by differences or ratios first.
- Power-rule first, no rate → average rate → shrink the interval → then the rule.
- Sample offered as population → "About the sample, or a claim about the population?"
- Waits for rescue → count three; sit on your hands.
- Stuck after a real try → one think-aloud, then a parallel item.
- Session is only worksheets → put one item on a graph, table, or diagram; mix yesterday.
- Parent feels rusty and reaches for the pencil → put the representation back; let them build and say.[22]

Revisit yesterday. Mix last week. Stop. That is the habit. It is not a diploma.

The talk-box shape, once

Later chapters will fill this week's opening question. They will not reprint this shape. When a chapter prints a talk box, it means this.

[22] Erin A. Maloney et al., "Intergenerational Effects of Parents' Math Anxiety on Children's Math Achievement and Anxiety," *Psychological Science* 26, no. 9 (2015): 1480–88. School homework-help study, not a homeschool trial. The portable caution: let the student build and say.

Opening question. Authentic. About linear versus quadratic structure, given / goal / bridge in a proof, function family from a table, unit-circle location and sign, average rate becoming instantaneous (with units), or sample versus population. No answer you have already written. Not “did you have fun” and not “what is the vocabulary word.” Locked openings live in the teaching chapters.

Three to five follow-ups. Pick; you do not need all five every Tuesday. Meaning: What does that number mean? Representation: Show me on a graph, table, diagram, unit circle, or rate sketch. Type or check: Is this linear or quadratic? Does that answer make sense? Mind-change: You may change your mind. Try it another way. Written reason or units: Say the reason out loud, or name the units of the rate.

How to wait. After you ask, a slow three. After they stop, wait again. If they are mid-reason, do not cut them off. Look at the graph, the diagram, or the table, not at the student, if the silence is hard.

What a stuck silence usually means. The question was vague. Wait-time after the question was zero. They are hunting a keyword. They reached for a formula with no representation. They think “=” means “the answer comes next.” They want the regression button or the calculator first. The numbers or the structure are too new. Pose, wait, and point back at the representation. You do not stack five questions, switch to a live controversy, or score the student with a percentile.

The in-home try-it shape, once

Later chapters will fill named types. They will not reprint this spine. When a chapter says “run the in-home try-it,” it means this.

Time. A medium sit, then stop. Fifteen minutes with a car-loan amortization row. Twenty with function-family sort cards. Pedagogical design, not a national minutes table.

Materials. Household objects: an amortization printout or spreadsheet the parent checks, scrap function cards, a blank unit circle, a rate table, a coin or spinner, proof paper. You do not need a kit.

Safety. Ordinary kitchen. Tone: curious, not courtroom. A car-loan row is a math object — not a sermon about debt culture. Pharmacy dosage as rate is a composite illustration — not medical advice.

The fun. Reading month-1 interest versus principal. Sorting families without a button. Placing $\pi/3$ on the circle. Estimating a limit from a table. Finger-tracing area under a speed graph.

The skill. Structure. Reasons. Family ID. Unit-circle meaning. Rate and accumulation. Sample versus population.

Types you will meet, not a lesson bank: car-loan APR amortization; pharmacy dosage as rate (composite); function-family sort; unit-circle dinner-table; limit from a table; speedometer accumulation; stats inference talk; same-to-both-sides repair; proof format swap.

The out-of-home try-it shape, once

"Outside" means the walk, the bridge or building, the sports sideline, the map, the ramp or stairs, the store or paycheck, the library or museum exhibit. Later chapters will fill named types. They will not reprint this spine.

Types you will meet: bridge or building angles; map trig or bearing; sports rates; physics-lab slope; store or paycheck percent; ramp slope as rise/run and as angle; library or museum exhibit data.

Safety, once. Roads, water, aisle courtesy. No shoplifting a "test." No lecture to a stranger's adolescent. No public quiz of a cashier. No climbing for a "trig photo." No stock tip. No medical advice from a dosage schema. A price tag is a number. Sports stats are rates. They are not identity fights.

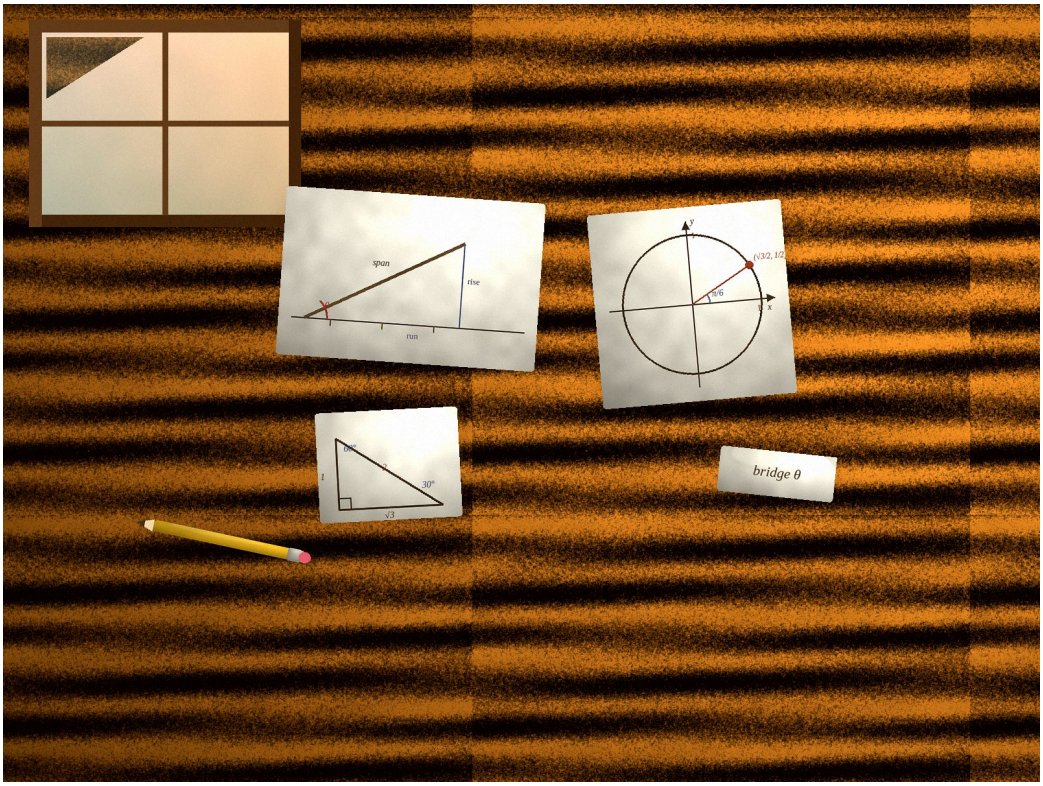

Figure 3: A marked bridge-angle sketch beside a unit circle

This week, said plainly

One focus skill — quadratic meaning; a short congruence proof; exponential versus linear from tables; unit-circle special angles; derivative as rate; or sample-versus-population talk. Four or five math hours of the session shape. Mixed review of last week's skill inside every hour. One in-home try-it and one out-of-home try-it. Friday: look at the week's exit tickets; pick one diagnostic wrong answer; decide whether next week repeats, narrows, or moves on. That is the honest answer to "lesson plans," not 180 days written in advance.

A short skill check is three objects, not a battery: an exit ticket of two to four items, one of them not today's new skill; one diagnostic wrong answer the student explains (letter-moving, formula without reason, keyword grab, regression-button family ID, power-rule-first, sample-as-population); a done-enough checklist by

course skill, not by birthday. Optional fourth: a publisher placement test after a gap — not a percentile, not an AP score as the only thing that counts.

The AI rules, once

Later chapters will point here in one sentence. They will not reprint this box. The student talks to the parent. The parent may use a tool. The student does not sit alone with an open chat as the only partner during the attempt when the parent is co-teacher. The math is on the table — function, graph, structure, proof move, free-response, labelled calculator or CAS. The student attempts first. The parent holds the answer key.

COPPA does not apply to ages 15–18. Name that once so you know why *Math for Young Minds* was stricter on account holding for ages 11–12. The lock is still pedagogical: parent co-holds / still in the room for math hour; no secret friend; unaided first; SCRIPT after the try; photo-to-key ban.

Ages 15–16 (COPPA off; parent still in the room — house default): parent co-holds the account. Scripts and parent–student talk. Not live open chat as the student's only partner during the attempt. Student may sit next to the parent while the parent uses a tool. They may hear a hint after they have tried. Labelled calculator or Desmos after attempt when the lesson calls for it.

Ages 17–18 (more agency; same bans; still not a secret friend): parent co-holds / can audit the account. Turning 17 or 18 is not a licence for a private math companion and not a licence for live-exam AI. The student may type their own "I think ___ because **.** ***We could check by*** ." The blanks are the student's. Labelled CAS, Desmos, GeoGebra, or Wolfram may be more frequent on calculator-allowed practice — still labelled, still not a substitute for no-calculator fluency, still not a chatbot finishing the free-response.

You may, on your account, after the student has tried:

- Explain this idea *to you* from a named lesson (title, page, CED topic, or today's object).
- Extra isomorphic practice with the answer key held by you. The student never sees the key. Do not photograph tonight's worksheet to generate "practice."

- A labelled SCRIPT after the try — a short spoken sequence for *you* to say, not a paragraph for the student to copy.
- A hint after an attempt (a question or named idea, not the finished number).
- Diagnose work the student already produced. Crop to the paper. Do not upload the student's face.
- Labelled tool after the try when the lesson matches exam rules: graphing calculator, Desmos, GeoGebra, Wolfram|Alpha, or CAS. Eyes open.

You may not:

- Paste the worksheet and ask the tool to complete it.
- Ask "what's the answer" or "just give me the steps" during the student's attempt, or let the student ask that of a window as the only partner.
- Photo-to-key. Point a camera at the page (Photomath, Mathway, Symbolab, or any scan-and-solve) so a solution pops up for the exact problem. Photo-to-answer is the ban.
- Park an unsupervised chatbot as the student's only partner during the attempt — not at 15, not at 18. No "math friend" account.
- Write the student's work. The model does not fill the free-response, the proof reasons, or "therefore $f'(x) =$ ___."
- Invent facts. Look.
- Grade from a detector score. Print a certificate of fluency.
- Hand the student an agent (Hermes Agent, Grok Bot, or any cloud-computer worker) as a math partner.
- Coach live-exam cheating. No chatbot or Photomath during a live AP exam or closed timed assessment.

In a high-school math field experiment with nearly 1,000 Turkish students in grades 9–11, an unguarded chatbot raised assisted practice about 48 percent and cut unaided exam grades about 17 percent; a guarded tutor that withheld the answer raised practice about 127 percent and left the unaided exam about the same as control (Bastani et al., 2025).[23] **That is this age band's math trial — not a U.S. home-**

[23]Hamsa Bastani et al., "Generative AI without guardrails can harm learning: Evidence from high school mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. Field experiment, nearly 1,000 Turkish high-school math students, grades 9–11:

school RCT. This is a useful study, not a promise that every home will see the same result. The model may prepare the adult and the next problem; it may not do the student's problem during the attempt.

Federal practice guides and the National Mathematics Advisory Panel name solved problems, structure, strategies, and mutually reinforcing understanding, fluency, and problem solving. They do not name a chatbot as the math teacher.[24]

Detectors mislabelled more than half of some human essays as machine-written (Liang et al., 2023: 61.22 percent false-positive on human TOEFL essays).[25] Do not police a sixteen-year-old's free-response with a score. Language models invent sources (Walters and Wilder, 2023: 55 percent of GPT-3.5 citations and 18 percent of GPT-4 citations fabricated in that study).[26] If you asked a tool for a paper, open the paper.

Names stay separate. Hermes 4 is a model. Hermes Agent is an agent. Grok is a chat assistant. Grok Bot is a cloud-computer agent. Game artist makes pictures. None of these is a 15–18 math class. There is no Grok Bot education SKU. Agents are not chatbots.

COPPA covers children under 13. It does not gate this band. The house still refuses a secret friend. Keep accounts, logs, and keys where the parent can open them.

Not a secret friend. A tool that talks in the first person is still a tool. It does not get a bedroom, a private channel, or a promise to keep secrets from the parent.

GPT Base +48% practice / −17% unaided; GPT Tutor +127% practice / unaided ≈ control. **ON-AGE for this book** — high-school math trial, not a U.S. homeschool RCT.

[24]Search of IES algebra and related practice guides and National Mathematics Advisory Panel, *Foundations for Success* (2008), for a recommendation to sit a student with a generative chatbot: it is not there.

[25]Weixin Liang et al., "GPT detectors are biased against non-native English writers," *Patterns* 4, no. 7 (2023): 100779: average false-positive rate on human TOEFL essays 61.22 percent.

[26]William H. Walters and Esther Isabelle Wilder, "Fabrication and errors in the bibliographic citations generated by ChatGPT," *Scientific Reports* 13 (2023): 14045: 55 percent of GPT-3.5 citations and 18 percent of GPT-4 citations were fabricated in that study. Photomath "a solution will pop up for your exact problem" sits next to "designed for learning" — the loop is the ban. Khanmigo "never gives you the answer" is a design goal on a product page, not a kitchen RCT.

> You can teach this entire book with no AI. The hour still has the same shape.

A Tuesday, said plainly

Here is an illustration, not a reported family. A parent has spent five minutes looking at a solved quadratic beside a U-shaped sketch. The student warms up on factoring a friendly trinomial. The parent shows one worked quadratic, think-aloud, table and graph visible, then puts the pencil down. The student tries. The parent asks, “If this is linear, what stays constant — the rate or the second difference?” and waits. The student reaches for the formula. The parent hears it, asks them to name the family from the table first, and they rebuild. Exit: “These two are yours. I won’t help.” On a seventeen-to-eighteen day the same shape holds with a unit-circle placement, or — for a ready student — a shrinking-interval rate with units.

You can run that hour. You need today’s idea, a sentence you refuse to finish, and the willingness to stop talking. Later chapters will say: run the session as in *The Math Hour*.

Chapter 1

Algebra I: linear, systems, quadratic with meaning

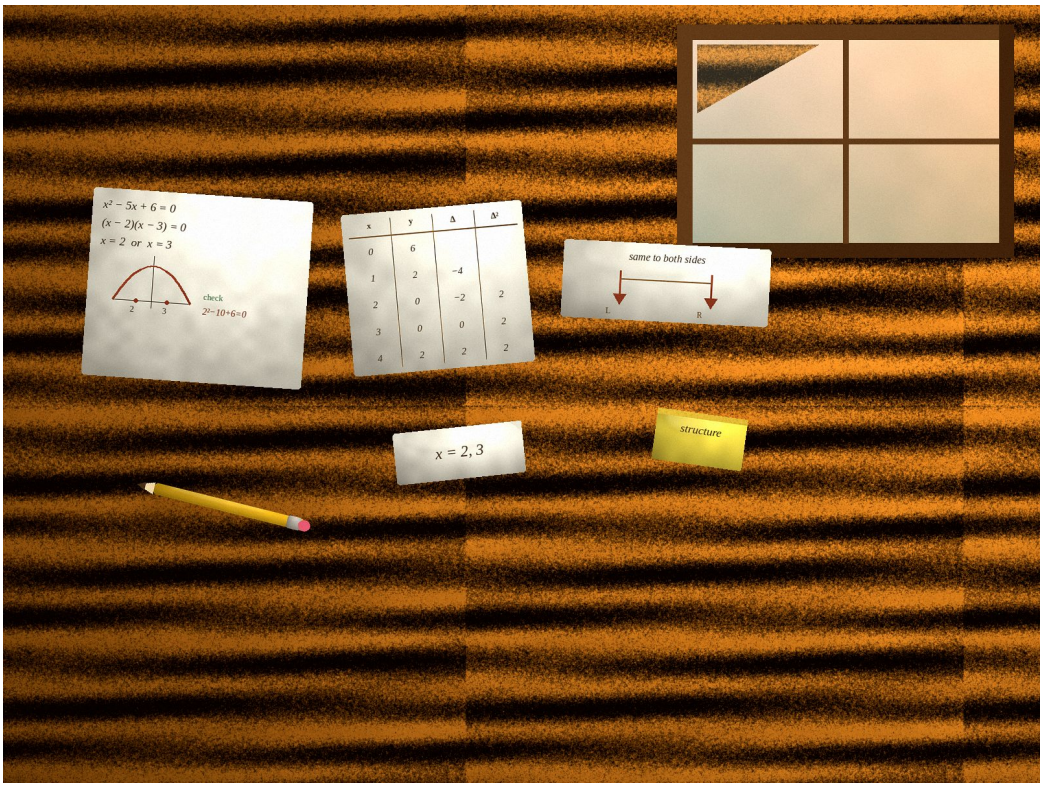

Figure 4: A kitchen-table still-life: scrap paper with a solved quadratic beside a U-shaped sketch; a small table of x-values with first and second differences; arrows labelled “same to both sides.” No people. No logos.

Why this matters

If this is linear, what stays constant — the rate or the second difference?

That question is the whole chapter in one breath. A linear relationship keeps a constant rate of change. A quadratic keeps a roughly constant second difference. When you know which kind of sameness you are looking at, the equation stops being a letter hunt and becomes a story about what stays the same.

Algebra I in this book is the first spine chapter of high-school mathematics — not an age-fourteen edge labelled only. That earlier grain lived in *Math for Young Minds*. One pointer: same-to-both-sides and early linear thinking continue here. Then we teach linear equations and inequalities, systems with schemas, and quadratic structure with meaning, at fifteen-to-eighteen grain. This book is not a reprint of *Mathematics for Homeschooling*, which compressed high-school algebra into survey chapters. One pointer, then teach this week's object. It is not a sequel to *Critical Thinking for Emerging Adults*. One pointer, then teach mathematics.

Many adults remember Algebra I as "move the 3 to the other side and flip the sign," then "plug into the quadratic formula." That speech can arrive later as informal talk for a legal both-sides move, or as a shortcut after a graph and a table have made the roots visible. It is a weak first story. The first story is: same operation, both sides; structure visible in a graph, a table, and an equivalent form; strategies chosen on purpose once some competence exists.

Jon Star and colleagues, in a practice guide for algebra in grades 6–12, recommend three moves that fit this table: study solved problems — including incorrect ones — so students analyze reasoning before they race blank worksheets; notice and use structure in algebraic representations; intentionally choose among alternative strategies and say why both are legal.[27] Those recommendations are on-age for Algebra I. They are a useful guide, not a promise that every home will see the same result, and not a franchise.

[27] Jon R. Star et al., *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students* (NCEE 2015-4010), April 2015. Recommendation 1: use solved problems to engage students in analyzing algebraic reasoning and strategies (minimal evidence), including incorrect solved problems. Recommendation 2: utilize structure of algebraic representations (minimal evidence). Recommendation 3: intentionally choose among alternative strategies (moderate evidence); flexibility is supported once some competence exists. Grades 6–12 — on-age for Algebra I. Access date for URLs in these notes: 5 September 2026.

What this idea unlocks is everything that follows. Geometry proofs assume you can keep claims true under change. Algebra II function families assume you can tell linear from quadratic from exponential without a regression button. Precalculus and calculus need rate language that already means something. If = still means "compute and write the blank," every new letter is a hunt. If both sides stay the same amount, and if a table can show which family you are in, the student can narrate legal moves, catch illegal ones, and check.

Why teach this *now*, at fifteen to eighteen? Because school algebra's major topics — symbols and expressions, linear equations, quadratic equations, functions — are the live agenda for this band.[28] Because some sixteen-year-olds still need a careful Algebra I restart without shame: federal parent-survey rows for grades 9–12 that year showed Algebra I and still-arithmetic sitting side by side as common offerings, not as a finished story for every birthday.[29] Because what is developmentally appropriate depends on prior opportunities to learn. Birthday is not a placement instrument.[30]

You do not need to be a mathematician. You do need to hear letter-moving with no structure as a missing-meaning move, not a cute slip; to hear the quadratic formula offered before a graph or a table; and to ask what stays constant, or what was done to both sides, without taking the pencil.

This week you can learn to hear "move and flip" as magic, formula-first with no representation, and first differences treated as if they were second differences. Today the student can solve a linear equation or a friendly quadratic narrating each legal move and checking, or sort a table as linear versus quadratic by differences.

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[28]National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (U.S. Department of Education, 2008), Major Topics of School Algebra; Finding 10 on mutually reinforcing conceptual understanding, computational fluency, and problem solving.

[29]U.S. Department of Education, National Center for Education Statistics, Parent and Family Involvement in Education survey rows for grades 9–12 subject offerings *that year* (PFI 2016 Table 9 pattern as copied in the research brief): Algebra I and still-arithmetic both appear as common high-school math rows — place by skill, not as a kitchen promise about your student.

[30]NMAP (2008), Finding 15: what is developmentally appropriate is largely contingent on prior opportunities to learn; age-stage "not ready" claims are wrong when prerequisites are present — and "old enough" does not create prerequisites. Critical Foundations continuity if unfinished (Finding 4 / Finding 6 grain).

For the parent: understand it yourself

Many adults feel rusty on high-school math they last saw as a student. That is ordinary. A diet of "get x alone" without saying how trains letter-chasing. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The student still writes the steps.

Everyday picture. A car ramp or a flight of stairs. Each step rises the same amount for the same run. That constant rise-over-run is a linear rate. Or a kitchen: three identical packs of crackers cost twenty-one dollars together. `3x = 21`. Divide both sides by 3. Each pack is seven. Check: three packs at seven is twenty-one. Kitchen and cars motivate. The written equation and the check still happen.

A second picture, for quadratic. A ball thrown upward, or a simple area story: length is two more than width, area is twenty-four. The table of height versus time does not climb by the same step each second. The *change in the change* settles into a pattern. That second difference is the everyday clue that you are not in a linear story anymore.

Precise picture. An equation is a claim that two expressions name the same value. Solving means finding the value (or values) that make the claim true by transforming the equation into an equivalent one — same truth — until the unknown is alone or the roots are visible. Every transforming step is an operation applied to both sides, or an equivalence that preserves truth. "Move the 5 and flip" is informal speech for subtracting 5 from both sides. Teach the both-sides story first. Informal speech can arrive later, named as a nickname for the legal move.

A **linear** relationship has a constant first difference when the input steps are equal. The rate stays constant. A **quadratic** relationship has first differences that themselves change at a roughly constant rate — a constant second difference on an equally spaced table. The second difference is not "the rate." The rate is changing. Roots are input values that make the output zero — where the graph meets the horizontal axis, or where the story's quantity is zero. Factored form makes roots visible. Standard form makes the y-intercept visible. A graph makes the vertex and the shape visible. Different forms carry different information.[31]

[31] Jon R. Star et al., *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students* (NCEE 2015-4010), April 2015. Recommendation 1: use solved problems to engage students in analyzing algebraic reasoning and strategies (minimal evidence), including incorrect solved problems. Recommendation 2: utilize structure of algebraic representations (minimal evidence). Recommendation 3: intentionally choose among alternative strategies (moderate evi-

A **system** is two (or more) claims that must be true together. Substitution, elimination, and graphing are strategies. Choose among them once some competence exists — Star notes that flexibility lands better after some procedural knowledge is in place.[32] Keyword theater ("altogether means add") is not a schema. Sort the story type first: two rates, two totals, a mixture, a comparison. Then write equations. Then solve. Then check in both original equations.

Study a solved problem before racing blanks. Here is a correct linear one, narrated:

`2x + 3 = 11`

Subtract 3 from both sides: `2x = 8`

Divide both sides by 2: `x = 4`

Check: `2(4) + 3 = 11`. True.

Here is an incorrect solved twin you should be able to hear — an autopsy, not a gotcha:

`2x + 3 = 11`

Someone writes: `2x = 11 − 3` (legal so far)

Then: `x = 8 − 2 = 6` — they "moved the 2" instead of dividing both sides by 2. The illegal step broke equality. The student's job is to find where the sameness broke.

Here is a quadratic with meaning beside the algebra:

`y = x^2 − x − 6`

Table for `x = −2, −1, 0, 1, 2, 3`: outputs `0, −4, −6, −6, −4, 0`. First differences: `−4, −2, 0, 2, 4`. Second differences: `2, 2, 2, 2` — roughly constant. Roots at `x = −2` and `x = 3`. Factored form `(x + 2)(x −`

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dence); flexibility is supported once some competence exists. Grades 6–12 — on-age for Algebra I. Access date for URLs in these notes: 5 September 2026.

[32] Jon R. Star et al., *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students* (NCEE 2015-4010), April 2015. Recommendation 1: use solved problems to engage students in analyzing algebraic reasoning and strategies (minimal evidence), including incorrect solved problems. Recommendation 2: utilize structure of algebraic representations (minimal evidence). Recommendation 3: intentionally choose among alternative strategies (moderate evidence); flexibility is supported once some competence exists. Grades 6–12 — on-age for Algebra I. Access date for URLs in these notes: 5 September 2026.

3) shows those roots. The graph is a U that crosses the axis at those two places. The formula can confirm; it should not be the first and only story this week.

Wrong answers you should be able to hear

1. "*Move the 3 to the other side and flip the sign," offered as the first and only story, with no both-sides narration.* The student may get lucky on one-step items and freeze when expressions sit on both sides, or when the undo is multiply/divide by a variable expression. Ask: "What did we do to both sides?" Put it on a balance sketch or two columns.

2. *Formula-first on a quadratic with no graph or table.* Roots arrive as numbers with no story. Ask: "Show me on the table and the graph. What do the roots mean here?"

3. *Treating first differences and second differences as the same thing.* Linear and quadratic collapse into one blur. Ask the locked opening: what stays constant — the rate or the second difference?

4. *Operating on only one side, or skipping the check by substitution.* Equality broke, or the check never happened. Point at the illegal line. Ask which step was not done to both sides. Check in the original.

5. *Keyword theater on a system story: "altogether means add," then equations that do not match the story; or freezing when variables appear on both sides.* Sort the schema first. Then write. Then solve. Then check in both equations.

A sixth you will also hear: "I'm sixteen, so Algebra I is done," while Foundations or linear/quadratic structure are still soft. Place by skill. An eighteen-year-old missing linear/quadratic structure does Algebra I repair without apology.[33]

Five-minute parent warm-up

Do this before the lesson, on a scrap of paper, no student in the room.

Minute 1. Solve `2x + 3 = 11` by same-to-both-sides. Narrate out loud. Check by substitution.

[33] NMAP (2008), Finding 15: what is developmentally appropriate is largely contingent on prior opportunities to learn; age-stage "not ready" claims are wrong when prerequisites are present — and "old enough" does not create prerequisites. Critical Foundations continuity if unfinished (Finding 4 / Finding 6 grain).

Minute 2. Make a tiny table for `y = 2x + 1` with `x = 0, 1, 2, 3`. Name the constant first difference. Say: “The rate stays constant. This is linear.”

Minute 3. Make a tiny table for `y = x^2` with `x = 0, 1, 2, 3, 4`. Compute first and second differences. Say: “The second difference stays roughly constant. This is quadratic.”

Minute 4. Look at a wrong solved problem you invent: someone “moves the 2” on `2x = 8` and gets `x = 6`. Mark the illegal step. You are installing the miss so you can hear it tomorrow.

Minute 5. Write the sentences you will actually say: “If this is linear, what stays constant — the rate or the second difference?” Under it: “What did we do to both sides? Show me on the table and the graph. Check.” Put the pencil down. Those sentences are the lesson.

If you can do those five minutes, you are ready to sit down. The student writes the steps. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version. Here is the shape scaled to this idea. Typical sit: forty-five to seventy-five minutes; stretch toward ninety if a system or a quadratic investigation needs it.

Warm-up (5–10 minutes, unaided). Two linear solves by same-to-both-sides the student already knows. One tiny table: is it linear? Paper. No device. No photo-to-key.

Short model (8–12 minutes). One solved problem with think-aloud — often a friendly quadratic with graph and table beside the algebra, or a two-equation system with the schema named. Include an incorrect twin for thirty to sixty seconds and mark the illegal step together. You talk for a few minutes. Then you stop.

Student attempt (15–30 minutes). One longer problem or a short set of *today’s type* — linear with variables on both sides, a system by substitution or elimination, or factor-and-roots with meaning. The student writes every line. The student narrates. You wait. Struggle before rescue: ask, wait, hint (“same to both sides — show me,” or “what do the second differences say?”), then a short model on *your*

scrap, not as their work.

One good question, then wait. Use the locked opening. A slow three. After they stop, wait again. Look at the table or the graph, not at the student's face, if the silence is hard. Stahl's think-time is a classroom convention about three seconds; Rowe's windows were science class.[34] Use the pause. Neither is a homeschool trial of algebra.

Mixed practice (10–15 minutes). Yesterday's linear next to today's quadratic. A system next to a single equation. Mixing is how the student learns *when* to use which move.

Exit ticket (3–5 minutes). Solve one item. One-sentence meaning (what a root means, or what the rate is). Check. One item from last week. Done-enough is right, or wrong-with-a-reason we can use tomorrow.

Exact wording you can say

"If this is linear, what stays constant — the rate or the second difference?"

"What did we do to both sides?"

"Show me on the table and the graph."

"What do the roots mean in this story?"

"Which form makes the vertex / intercepts visible?"

"Which step in this solved problem broke equality?"

"You may change your mind."

When they say "move the 3":

"Tell me the both-sides version of that move."

When they reach for the quadratic formula before any picture:

"Graph and table first. Then we can confirm with the formula."

When you are about to take over:

[34] Mary Budd Rowe, "Wait-Time and Rewards as Instructional Variables," NARST, April 1972, ERIC ED061103 — elementary science class, not a mathematics RCT. Robert J. Stahl, "Using Think-Time and Wait-Time Skillfully in the Classroom," ERIC Digest ED370885, 1994 — about three seconds as a convention.

"Your pencil. I'll wait."

Age-band moves: 15–16 / 17–18

15–16. Linear equations and inequalities with same-to-both-sides alive (including variables on both sides). Systems of two linear equations with a named schema — substitution and elimination as strategies, graph as a third view. Friendly quadratics: table, graph, factor when the numbers cooperate, roots with a one-sentence meaning. First vs second differences as the family test. Absolute value and inequalities: keep the both-sides story; when you multiply or divide by a negative, the inequality direction flips for a reason you can say, not as magic. If fraction/ratio Foundations are still soft, remediate without apology — birthday did not fix it.[35]

17–18. Same Algebra I objects if the skill gate is not clear yet — place by skill. If Algebra I structure is secure, deepen: messier systems, quadratic formula beside completing the square and graphing as chosen strategies, inequalities in two variables as a graph story, preparation for Algebra II function families in Chapter 3. Turning seventeen does not finish Algebra I by candlelight. An eighteen-year-old who still treats = as "the answer comes next" is still in the same-to-both-sides repair lane of this chapter.

A fifteen-year-old who can sort linear vs quadratic from a table and narrate a system check is not "too young" for this spine. A seventeen-year-old who cannot is not "behind on birthday calculus" — they are behind on Algebra I meaning.

First try-it for the student

On scrap paper:

Make a table for `y = 3x + 1` with four equal x-steps. Ask: "If this is linear, what stays constant — the rate or the second difference?"

Wait. If they name the constant first difference, ask them to try `y = x^2 − 4` next and name what stays roughly constant. If they reach for a formula, put the pencil on the differences column.

Later the same week, the autopsy:

[35] NMAP (2008), Finding 15: what is developmentally appropriate is largely contingent on prior opportunities to learn; age-stage "not ready" claims are wrong when prerequisites are present — and "old enough" does not create prerequisites. Critical Foundations continuity if unfinished (Finding 4 / Finding 6 grain).

Show a wrong solution to `2x + 4 = 10` that subtracts 4 from only one side, or a quadratic that "moves letters" into the formula with no table. Ask which step broke equality or skipped meaning. Do not scold. Name the miss. Repair with same-to-both-sides or with graph-and-table first.

How to fade help. First sitting: you build the table, you name the differences, they echo. Second: they fill the table, you wait, you hint ("first or second?"). Third: they sort two tables without your marks. Fourth: a wrong solved problem autopsy without your mark first. The quadratic formula, if it arrives this week at all, arrives after a graph and a factored form have shown the roots.

When to stop talking. When you hear yourself filling in the next line. When the sit has become a lecture titled *Properties of Equality* or *The Quadratic Formula*. When the student is mid-check and you have already asked a second and a third question. One good question beats five. Stop while they still have a table left in them.

This week's plan (honest "lesson plan"). One focus skill — for example, quadratic meaning from table and graph, or systems by substitution with a schema. Four or five sessions. One in-home try-it and one out-of-home try-it below. Friday: one diagnostic wrong solved problem plus the done-enough checklist. Not one hundred eighty worksheets. Not a percentile.

Practice that actually builds learning

Blocked, for a new move. The day you introduce second differences, only that family of tables, friendly numbers, graph visible. The day you introduce a wrong-solved autopsy, only autopsy. Mixing too early makes the student hunt for "the trick."

Mixed, for when to use it. Later the same week, a linear next to a quadratic next to a system. Mixing is the practice of choosing: constant rate, changing rate with constant second difference, or two claims together?

One incorrect example to diagnose. Letter-moving with no both-sides story; formula-first with no graph; first and second differences treated as the same; key-word grab on a system. Hear the missing meaning. The repair is a better question, then the representation, then — if needed — a named fact: "Same operation to both sides," or "Second differences roughly constant means quadratic family." Silence

in the face of a keyword error is not kindness.

These are illustrations, not reported families.

Named try-it: Wrong solved problem autopsy (in-home)

Time. Ten to fifteen minutes. Then stop.

Materials. One incorrect worked quadratic or system you wrote ahead (or last night's real miss, cropped to the math).

Safety. Tone is curious, not courtroom. No shaming. No screenshot of a classmate.

The fun. Finding the illegal step. Detective work.

The skill. Star's solved-problem habit; structure talk; relational =.

Offer a wrong solution. Ask which step broke equality or skipped the graph. Have the student rewrite the legal version and check. Do this once or twice a week while Algebra I meaning is new — not as a daily humiliation.

Named try-it: Same-to-both-sides repair (in-home)

Time. Ten to fifteen minutes.

Materials. The learner's own wrong rearrange; scrap paper; optional balance sketch.

Safety. Ordinary table. The point is legal moves, not a fight about intelligence.

The fun. Keeping equality true on purpose.

The skill. Refuse move-and-flip as magic; name the property; check.

Rewrite the illegal line as balance steps. Every line named: "I subtracted 3 from both sides." Check by substitution into the original. If absolute value or an inequality is in the room, keep the same discipline — flip the inequality direction only when multiplying or dividing by a negative, and say why.

Named try-it: Quadratic meaning table + graph (in-home)

Time. Fifteen to twenty minutes.

Materials. Scrap paper; equal x-steps; optional Desmos *after* the hand table, labelled.

Safety. Eyes open on the student's marks first. A graphing window after the try is a check, not a substitute for differences.

The fun. Watching second differences settle into a pattern.

The skill. Representation before formula; family ID; roots with meaning.

Build a table for a friendly quadratic. Compute first and second differences. Sketch the U. Mark the roots. Say what the roots mean in a one-sentence story (when height is zero; when profit is zero — composite illustration, not a business plan). Then, if you choose, confirm with factoring or the formula.

Named try-it: Ramp or stairs as linear rate (out-of-home)

Time. Ten to fifteen minutes looking; short write-up at the table.

Materials. Eyes; optional tape measure; scrap.

Safety. No climbing stunts. Sidewalk, park ramp, or stairs you already use. Roads. Ordinary courtesy.

The fun. "How steep?"

The skill. Slope as rate; constant first difference when run is equal; bridge toward systems stories about two rates.

Measure or estimate rise and run for one ramp. Write rise/run as a rate. Ask whether equal horizontal steps would give equal vertical steps. Hand full similar-triangle geometry to Chapter 2; hand function-family depth to Chapter 3. Today: linear rate with meaning.

Named try-it: Store or paycheck two-step story → system schema (out-of-home → table)

Time. Five minutes collecting numbers outside; fifteen to twenty minutes solving at home.

Materials. A receipt, shelf prices, or a pay stub percent the family already understands; scrap at the table.

Safety. Ordinary aisle courtesy. No public quiz of a cashier. No shoplifting a "test." Paycheck percent is a math object, not a sermon about workplaces. Tax-then-tip only if percent meaning already works; otherwise keep the story additive.

The fun. "We saw it outside — now it becomes two claims."

The skill. Schema not keyword; write two equations; solve; check by substitution in both.

Example shape (illustration, not a reported trip): two packages with different per-item costs and a known total; or hours at two rates that add to a paycheck total. Write the system at home. Choose substitution or elimination on purpose. Check. If the student grabs "altogether means add" and writes nonsense, sort the schema again before more algebra.

Talk box

Opening question (locked): "If this is linear, what stays constant — the rate or the second difference?"

Quadratic follow-ups (in this box)

1. For `y = ax^2 + bx + c`, what does `a` do to the graph?
2. What do the roots mean in this story?
3. Show me on the table and the graph.
4. Which form makes the vertex / intercepts visible?
5. You may change your mind. Try it another way.

How to wait

Ask. Count a slow three in your head. Look at the table or the graph, not at the student's face, if the silence is hard. After they stop talking, wait again. If they are

mid-reason, do not cut them off. Three seconds is a convention, not a sacrament.

What a stuck silence usually means

They reached for a formula with no representation. Or they think = means "the answer comes next." Or they offered "move and flip" as magic. Or wait-time after the question was zero. Or they are hunting a keyword on a system story. Or the numbers or the structure are too new. Or they are guessing what you want. Next move: smaller numbers, table first, wait, point back at differences or both sides. "Show me the first differences" is a hint. "x equals 4" said by you is not a hint; it is the grab. Done enough this week: a spoken narration. "First differences stay at 3 — linear. Check: plug back in."

For the student

This page is for you.

If this is linear, what stays constant — the rate or the second difference?

A linear story keeps the same rate when the input steps are equal. A quadratic story keeps a roughly constant *second* difference — the change in the change. That is how you know the family without pressing a regression button on a calculator.

An equation says two sides are the same. When you change one side, you change the other the same way, or the sameness breaks.

`2x + 3 = 11` means twice a number, plus three, is the same as eleven.

Subtract three from both sides. Divide both sides by two. Now `x = 4`.

Check: put 4 back in. `2(4) + 3 = 11`. True. If the check fails, something broke along the way. Find the step.

"Move the 3 and flip the sign" is a nickname some people use later. The real move is: same subtraction, both sides. Learn that story first.

A tiny worked example

Table for `y = x^2 − 4` with `x = −2, −1, 0, 1, 2`:

Outputs: `0, −3, −4, −3, 0`.

First differences: `−3, −1, 1, 3`.

Second differences: `2, 2, 2` — roughly constant. Quadratic family.

Roots where `y = 0`: `x = −2` and `x = 2`. Factored form `(x − 2)(x + 2)` shows them. The graph crosses the axis at those points. The formula can confirm later. Meaning comes first.

Someone "moves letters" into a formula with no table and cannot say what the roots mean. That skipped the story.

Two tries

1. Solve `3x − 5 = 10`. Write every line. Say what you did to both sides. Check.
2. Make a four-row table for `y = 2x + 3` and for `y = x^2`. Name which is linear and which is quadratic by differences. Or, if you are ready, solve a friendly system your parent sets and check in both equations.

Explain it back

Tell someone at the table what stays constant in a linear table versus a quadratic table. Show `2x + 3 = 11` as same-to-both-sides. Say what "move and flip" is nicknaming, when it is legal at all.

Challenge

Someone always isolates the letter without checking. What question do you ask them? Someone treats every curved graph as "just use the formula" with no table. How do you show second differences first?

You are allowed to struggle. You may use a table, a sketch, scrap paper. You write. You talk. If you get stuck, ask for a hint — not the finished number. Then try again.

When you talk, a sentence about the rate, the second difference, or both sides is enough for today. You do not have to call yourself finished with Algebra I because of a birthday. Today you solve, narrate, check, and sort a family.

A picture a computer made of a perfect parabola is a scene. It is interesting. It is not the table on your scrap.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The student treats = as "the answer comes next," or solves by letter-moving with no both-sides narration, or skips the check.

The operational view is doing the work of the relation. Next move: true/false sentences with expressions on both sides. Balance or two-column sketch only, for several days. Autopsy one wrong solved problem. Check by substitution every time for a week of short sits. Stay here if this is still the bottleneck. Quadratic worksheets will not hold on top of a green-light equal sign. A human who will sit with both sides and wait, not a solver app, is a reasonable next step if you have tried the both-sides work and the operational reading is still the whole hour.

2. The student reaches for the quadratic formula (or a regression button) before any table or graph, or treats first and second differences as the same.

Procedure arrived without family meaning. Next move: tables only for several sits. Linear vs quadratic sort cards. Name what stays constant out loud before any formula. Go ahead once they can sort two tables and explain roots on a graph in one sentence — then add factoring or the formula as a chosen strategy. A tutor is useful if formula-first remains the default after a couple of weeks of daily representation work *and* the hour has become a fight.

3. The student freezes on systems or word equations, or every sitting ends in a shrug, and the silence after your question is a wall.

The wait was zero, or they are hunting a keyword, or the schema is unclear, or the arithmetic inside the algebra is too new. Next move: shorter sits, numbers they already compute, the opening question from the talk box, wait a slow three twice. Sort the story type before writing equations. If silence is hard, you look at the table, not at them. If you hear yourself writing the next line, you have started doing the work. If Foundations (fractions, ratio) are soft, remediate those without calling it a birthday failure. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that fills the silence, is the release valve.

When to slow down. Operational = still in the room. One-sided moves still the default. Checks skipped. Formula-first every time. Sits so long that narration never starts. Those are brakes. "Not ready" because of age is the brake this book

will not use. “Not ready” because equality is not yet preserved on purpose, or because linear and quadratic still blur, is a real brake. An Algebra I workbook cover is not a reason to skip the table.

When to go ahead. The student can solve a linear equation narrating each legal move and checking; can sort linear vs quadratic from a table; can explain roots in one sentence on a friendly quadratic; and, when systems are in the week’s focus, can check a solution in both original equations. A wrong solved problem has been autopsied. Short sits are ordinary. Then geometry with reasons (Chapter 2) and function families (Chapter 3) have somewhere to sit. Being “good at algebra” because a student is fast at moving letters is not a reason to skip structure.

When to get a human tutor. You have run the table, or the autopsy, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear “move the 3” as a missing both-sides story. Outsourcing the hearing is the thing to avoid, not asking for help.

Correct an illegal step without crushing the attempt. “I hear move-and-flip. Show me that as a both-sides move.” Or: “I hear the formula. Show me the table first.” Then look. Hearing a wrong answer, asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The student attempts first. You hold the equation, the table, and the question. A tool may explain today’s idea *to you* from this chapter, suggest extra isomorphic problems you then vet (answer key held by you), write a short SCRIPT for *you* to say after the student has tried, offer a hint after an attempt, or help you diagnose work the student already produced. Crop to the paper. Do not upload the student’s face. The student never sees the key.

Ages 15–16: parent co-holds the account; parent still in the room for math hour. Ages 17–18: more agency; same bans; still not a secret friend during the attempt. COPPA does not apply at this age band — named once in the math hour — but the

pedagogical lock remains. Unaided first. SCRIPT after the try. Labelled calculator or Desmos after the attempt when the lesson calls for a graph check — eyes open, not a substitute for hand tables.

Leave these out of the hour: a chatbot as the only partner during the attempt; a tool that completes the worksheet; a camera pointed at the page so a solution pops up for the exact problem; a detector score; a certificate of Algebra I fluency; a cloud agent as a math partner. Photo-to-answer is the ban.

A language model will happily "move the 3" or dump the quadratic formula. Treat every model-supplied solution as untrusted until the student has narrated structure and checked. In a high-school math field experiment (grades 9–11), an unguarded chatbot made practice look better and left students worse when the window was closed; a guarded tutor that withheld the answer left the unaided test about the same as no AI.[36] That is this age band's math trial — not a U.S. homeschool RCT. This is a useful study, not a promise that every home will see the same result. The kitchen-table rule it supports: the model may prepare the adult and the next problem; it may not do the student's problem during the attempt.

A balance sketch, a hand table, scrap paper, and a wrong solved problem you wrote yourself are tools too. Use them, then fade them.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 9 Algebra I workbook" is a publisher's scope, not a legal grade, and not a transcript line. The record, when you need one, is a dated notebook, the task named, plus an exit ticket, plus one diagnostic item. The title a stranger can read is **Algebra I** — only if the year's work was that course. It is not Emerging Adults Math I. It is not a brand name as the credit.

Checklist before moving on

[36] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without guardrails can harm learning: Evidence from high school mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. Turkish grades 9–11 field experiment; unguarded chatbot +48% practice / −17% unaided exam; guarded tutor +127% practice / unaided ≈ control. ON-AGE for this book — not a U.S. homeschool RCT. Kitchen-table rule: the model may prepare the adult and the next problem; it may not do the student's problem during the attempt.

- You can hear "move the 3" with no both-sides story, and you can ask what was done to both sides.
- You can hear formula-first with no graph or table, and you can ask what stays constant — the rate or the second difference.
- The student can solve a multi-step linear equation, narrating each legal move, and check by substitution.
- The student can sort a friendly table as linear vs quadratic by first and second differences.
- Given a friendly quadratic, the student produces a graph sketch or accurate table and explains roots in one sentence — factoring or formula as a chosen strategy, not the only story.
- When systems are in the year's work: the student identifies a schema, solves by a named method, and checks in both original equations.
- At least one wrong solved problem has been autopsied: the illegal step named and repaired.
- At least one in-home try-it (table, autopsy, or same-to-both-sides repair) and one out-of-home try-it (ramp rate or store/paycheck schema) landed on paper.
- Short sits are ordinary. A spoken narration plus a check is enough. You did not require a full Algebra II polynomial dump as proof of Algebra I.
- You can hear operational =, one-sided moves, keyword theater, formula-first, and "birthday finished Algebra I," and you can ask a good question, without taking the pencil.

If most of that list is true, go on to geometry with reasons, even if the birthday says otherwise. If the birthday says "Algebra II" and linear/quadratic structure is still soft, stay. Chapter 2 asks for given, goal, and bridge. It needs same-to-both-sides and representation habits underneath. Chapter 3 asks for function families without a regression button. It needs this chapter's difference test.

A path through high-school mathematics is the promise. A diploma is not. A percentile is not. An AP score is not the only path that counts.

Chapter 2

Geometry with reasons

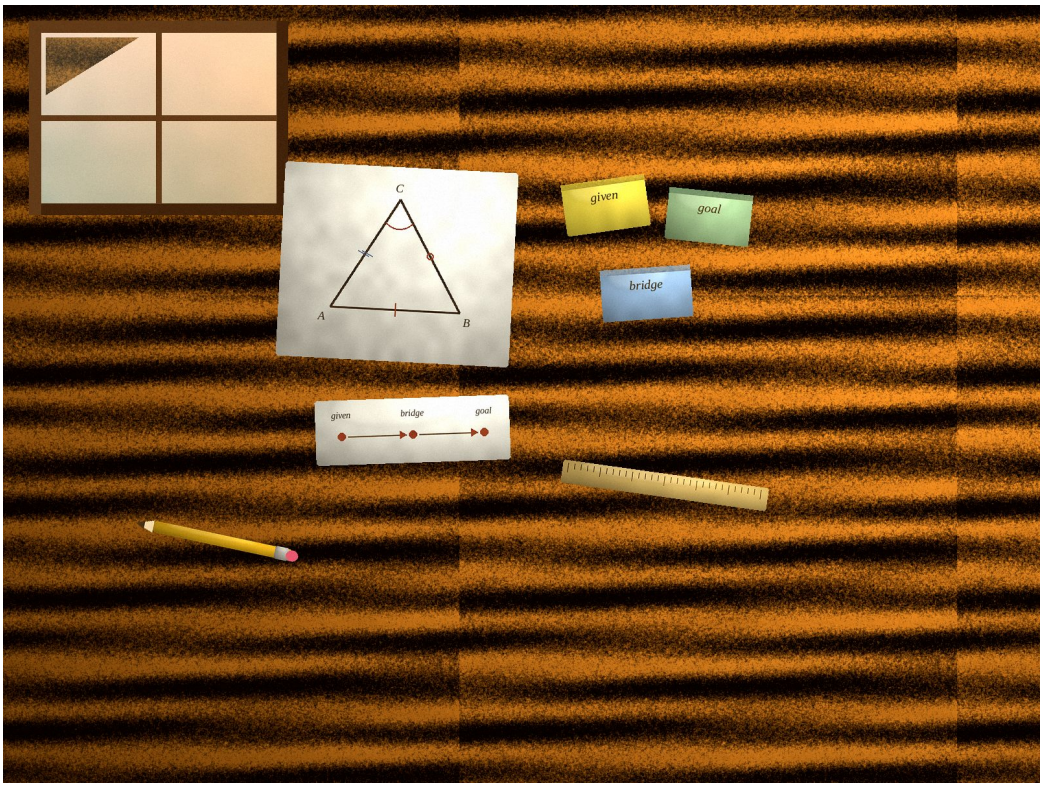

Figure 5: A kitchen-table still-life: a marked triangle diagram with sticky notes "given," "goal," and "bridge"; light flow arrows from premises to conclusion. No people. No logos.

Why this matters

What are we given? What must we show? What bridge gets us there?

That triad is the whole chapter in one breath. Geometry with meaning is not a laminated formula sheet. It is a habit of saying what is true, what must follow, and which definition, postulate, or theorem carries you across. When those three are clear, the format — two-column, paragraph, or flow — is a choice about how to write the argument, not a religion about what counts as real mathematics.

Many adults remember high-school geometry as "memorize SAS, ASA, SSS, then fill two columns until the teacher is satisfied," or as "plug into area and volume formulas." Formulas can appear later as summaries of reasoned measurement. Two-column can be a useful scaffold. Neither is the first story. The first story is: mark the figure; name the givens; name the goal; find a bridge; say a reason out loud; then pick a format once the logic is clear.

Michelle Cirillo and Patricio Herbst, writing about authentic proof practices, press a point that fits this table: focus on logical arguments rather than on the form of proof used; let students take a greater role in proving; use multiple proof representations.[37] The National Council of Teachers of Mathematics, in its geometry essential concepts, says proof demonstrates truth or falsity and can be communicated in a variety of ways — two-column, paragraph, and more — and that proofs may use transformations, coordinates, or algebra.[38] Two-column proving became a sticky American school custom; custom is not mathematical necessity.[39] Those sources are a useful grain, not a promise that every home will see the same result, and not a rating of your living room.

What this idea unlocks is honest high-school geometry and cleaner algebra later. Coordinate proofs reuse Algebra I structure. Right-triangle trigonometry needs

[37] Michelle Cirillo and Patricio G. Herbst, "Moving Toward More Authentic Proof Practices in Geometry," *The Mathematics Educator* 21, no. 2 (2012): 11–33. Emphasis on logical arguments rather than form; multiple proof representations; greater student role in proving. Access date for URLs in these notes: 5 September 2026.

[38] National Council of Teachers of Mathematics, *Catalyzing Change in High School Mathematics* essential concepts for geometry and measurement (GARP.1–3 grain): proof communicates in a variety of ways; proofs may use transformations, coordinates, or algebra. Used here as a professional map, not a homeschool statute.

[39] Historical stickiness of two-column proving in American school geometry is discussed via Cirillo & Herbst (2012) citing Herbst's historical work (2002). Custom is not mathematical necessity. Full independent page-quotes from Herbst 2002 were not relied on beyond that attribution chain.

reasons about similar triangles before SOHCAHTOA becomes a chant. Precalculus and calculus inherit diagrams that already mean something. If geometry is only a formula card, every new figure is a hunt for the right blank. If given, goal, and bridge are spoken first, the student can change formats without losing the argument.

Why teach this *now*, at fifteen to eighteen? Because proof and reasons belong in the spine from early weeks — waiting for a single "proof unit" after months of formulas trains the wrong habit. Because conceptual understanding, fluency, and problem solving belong together; formula-only fails that test.[40] Because ages 15–16 can write short congruence and similarity arguments with reasons, use coordinate geometry as a proof tool, and meet right-triangle trig as an edge; ages 17–18 can strengthen proof writing and lean more on transformation and coordinate approaches — place by skill, not birthday.[41]

This book is not a reprint of *Math for Young Minds*, which owned middle-grades geometry-with-reasons at a lighter grain. One pointer: reasons still matter. Then we teach high-school proof options and figure habits. It is not a reprint of *Mathematics for Homeschooling*. One pointer, then teach this week's object.

You do not need to be a mathematician. You do need to hear a formula without a reason as a missing-argument move, not a cute slip, and to ask what is given, what must be shown, and what bridge gets there, without taking the pencil.

This week you can learn to hear column-filling with no global argument, "SAS means congruent" with no corresponding parts named, and panic when a figure is rotated. Today the student can mark givens, state a bridge aloud, and write a short proof in any accepted format with reasons.

For the parent: understand it yourself

Many adults feel rusty on high-school geometry they last saw as a student. That is ordinary. A diet of formula sheets without "why" trains blank-filling. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The student still writes the reasons.

[40] National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 10: conceptual understanding, computational fluency, and problem solving are mutually reinforcing.

[41] NMAP (2008), Finding 15: place by skill; prior opportunities to learn matter more than age-stage slogans.

Everyday picture. A picture frame that must be square. You measure the diagonals. If they are equal (and the sides behave), you have a reason to trust the corner — not because a sticker said "square," but because a property held. Or a ramp: the rise and run make a right triangle similar to a steeper model triangle. The angle of inclination is not a magic number from a card; it follows from trig ratios once the right triangle is identified.

A second kitchen-table picture. Two triangles drawn on scrap. You mark tick marks on equal sides and arcs on equal angles. Your finger traces which vertex matches which. Congruence is a claim about corresponding parts. Saying "SAS" without pointing is like saying "because algebra" without a step.

Precise picture. A **proof** is a connected sequence of statements, each justified by a definition, a given, a previously established result, or a valid inference, showing that the goal follows from the givens. The **bridge** is the main idea that connects them — often a congruence criterion, a similarity criterion, a parallel-line angle relationship, a coordinate calculation, or a transformation that preserves the needed property.

Three formats, honest tradeoffs:

Format	Helps	Watch for
Two-column	Forces statement + reason pairing	Filling columns without a global argument
Paragraph	Closer to mathematical writing	Omitting reasons; jumping to the conclusion
Flow-chart	Shows premise → conclusion arrows; good planning	Less familiar in some books; may need a later rewrite into paragraph or two-column

Start hard proofs with a know / need / bridge outline or a flow. Let the learner pick two-column or paragraph for the write-up once the logic is clear. Demand a reason either way.[42] [43]

[42] Michelle Cirillo and Patricio G. Herbst, "Moving Toward More Authentic Proof Practices in Geometry," *The Mathematics Educator* 21, no. 2 (2012): 11–33. Emphasis on logical arguments rather than form; multiple proof representations; greater student role in proving. Access date for URLs in these notes: 5 September 2026.

[43] National Council of Teachers of Mathematics, *Catalyzing Change in High School Mathemat-*

Coordinate geometry is a proof tool: place a figure on axes, compute distances or slopes, and conclude with algebra you already trust from Chapter 1. Example grain: to show a quadrilateral is a parallelogram, you might show both pairs of opposite sides have equal length (distance formula) or that both pairs of opposite sides have equal slope (parallel). The algebra is the reason-carrier. The figure still gets marked.

Transformations (reflection, rotation, translation) can prove congruence when the motion maps one figure onto the other and preserves the needed measures. You do not need a software package to say: "Reflect across this line; the image matches, so corresponding lengths match." Software can explore after the try; the reason still gets spoken.

Right-triangle trigonometry is an edge in this chapter: once similar right triangles are justified, sine, cosine, and tangent become named ratios — not the entire trig course (Chapter 4 owns the unit circle). If the student chants SOHCAHTOA at a non-right triangle, stop and rebuild the right triangle first.

Wrong answers you should be able to hear

1. *Reciting a formula with no reason — area, volume, or Pythagorean theorem as a blank to fill.* Ask: "What must be true about this figure first? How do you know?"
2. *Filling two-column without a global argument — rows that look busy but do not connect givens to goal.* Ask the locked opening. Have them say the bridge before another row.
3. *"SAS means congruent" without saying which parts correspond, or mixing corresponding parts when the figure is rotated.* Put a finger on the figure. Name the vertices in order.
4. *Treating two-column as the only valid proof, and refusing paragraph or flow as "not real."* Formats are options. Reasons are required. Show the same argument in a second format once this week.
5. *Panic when the figure is rotated or reflected, or grabbing SOHCAHTOA with no right triangle identified.* Redraw; mark the right angle; name the

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ics essential concepts for geometry and measurement (GARP.1–3 grain): proof communicates in a variety of ways; proofs may use transformations, coordinates, or algebra. Used here as a professional map, not a homeschool statute.

similar triangles or the ratios after the geometry is clear.

A sixth you will also hear: waiting until "the proof unit" in April while autumn was only formulas. Bring short proofs early. A seventh: "I'm sixteen, so Geometry is done," while reasons are still absent. Place by skill.

Five-minute parent warm-up

Do this before the lesson, on a scrap of paper, no student in the room.

Minute 1. Draw two triangles that share a side. Mark two pairs of congruent sides and the included angles. Say out loud: "Given… Goal: triangles congruent. Bridge: SAS on these corresponding parts." Point with your finger.

Minute 2. Write the same argument as three flow boxes: premises $\rightarrow$ SAS $\rightarrow$ congruence. Then rewrite as two sentences of paragraph proof with reasons named.

Minute 3. Rotate the figure 90 degrees on the page. Rematch corresponding parts out loud. Install the miss you will hear tomorrow: someone who says SAS while pointing at the wrong angle.

Minute 4. Sketch a ramp as a right triangle. Label opposite, adjacent, hypotenuse for one acute angle. Say: "Tangent is opposite over adjacent — after I know it is a right triangle."

Minute 5. Write the sentences you will actually say: "What are we given? What must we show? What bridge gets us there?" Under it: "Say the reason out loud before you write the format." Put the pencil down. Those sentences are the lesson.

If you can do those five minutes, you are ready to sit down. The student writes the reasons. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version. Here is the shape scaled to this idea. Typical sit: forty-five to seventy-five minutes; stretch toward ninety when a proof needs room.

Warm-up (5–10 minutes, unaided). Mark congruent parts on a figure already drawn. Name one pair of vertical angles or corresponding angles from a parallel-line sketch. Paper. No device. No photo-to-key.

Short model (8–12 minutes). Know / need / bridge on one figure, then a flow draft. Include thirty seconds on a formula-only wrong twin: someone who wrote an area number with no units and no reason the figure allowed that formula. You talk for a few minutes. Then you stop.

Student attempt (15–30 minutes). Transpose the flow into paragraph or two-column — student's choice once the logic is clear — or write a short congruence proof from a new figure. Mark the diagram first. You wait.

One good question, then wait. Use the locked opening. A slow three. Look at the figure, not at the student's face, if the silence is hard. Stahl's think-time is a classroom convention; Rowe's windows were science class.[44] Use the pause.

Mixed practice (10–15 minutes). An angle chase next to a short proof. A coordinate distance next to a congruence claim. Mixing is how the student learns *when* a calculation is a reason and when a criterion is the bridge.

Exit ticket (3–5 minutes). "Which step used vertical angles?" or "Name the bridge in one sentence." One quick mark-the-figure item. Done-enough is right, or wrong-with-a-reason we can use tomorrow.

Exact wording you can say

"What are we given? What must we show? What bridge gets us there?"

"Say the reason out loud before you write the format."

"Which parts correspond? Show me on the figure with your finger."

"Pick a format once the logic is clear."

"You may change your mind."

When they recite a formula with no reason:

"What must be true first — and how do you know?"

When they fill columns without a bridge:

"Stop writing rows. Say the bridge in one sentence."

When you are about to take over:

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[44] Mary Budd Rowe, wait-time in elementary science (1972) — not a mathematics RCT. Robert J. Stahl, think-time ~3 seconds as a classroom convention (1994 ERIC digest).

"Your pencil. I'll wait."

Age-band moves: 15–16 / 17–18

15–16. Congruence and similarity with reasons; triangle theorems (sum of angles, isosceles base angles, Pythagorean theorem as a justified right-triangle result); parallel lines and transversals with named angle relationships; coordinate geometry as a proof tool (distance, midpoint, slope for parallel/perpendicular); area and volume as justified measurement with units; right-triangle trigonometry as an edge once similar right triangles are clear. Short proofs early and often. Formats optional; reasons required.

17–18. Stronger proof writing — longer chains, clearer paragraph proofs, more deliberate format choice. More transformation and coordinate approaches alongside synthetic arguments. If Geometry reasons are still soft, stay in this chapter's spine without apology. If reasons are secure, deepen toward precalculus readiness (Chapter 4) while keeping proof habits alive. Turning eighteen does not finish Geometry by candlelight.

A fifteen-year-old who can state given, goal, and bridge and write a short proof with reasons is not "too young." A seventeen-year-old who only has formulas is not finished with this chapter.

First try-it for the student

On scrap paper:

Mark a figure with givens. Ask: "What are we given? What must we show? What bridge gets us there?"

Wait. If they jump to two-column, ask for the bridge in one spoken sentence first. If they say a criterion with no pointing, ask them to show corresponding parts with a finger.

Later the same week, the format swap:

Same triangle proof — flow first, then paragraph or two-column. Compare: did any reason disappear in the rewrite?

How to fade help. First sitting: you mark the figure, you say know/need/bridge, they echo. Second: they mark, you wait, you hint ("what's the bridge?"). Third: they choose a format and write reasons without your outline first. Fourth: read a finished proof together and ask which step used a named theorem.

When to stop talking. When you hear yourself dictating column two. When the sit has become a lecture titled *List of Postulates*. When the student is mid-reason and you stack a second and third question. One good question beats five. Stop while they still have a figure left in them.

This week's plan (honest "lesson plan"). One focus skill — for example, triangle congruence reason → short proof in a chosen format. Four or five sessions. One in-home try-it and one out-of-home try-it below. Friday: read a finished proof; exit on "which step used…?"; out-of-home bridge angle if weather and safety allow. Not one hundred eighty worksheets. Not a percentile.

Sample week (triangle congruence → short proof).

Monday. Warm-up: mark congruent parts. Model: know/need/bridge → flow. Attempt: complete the outline on a twin figure. Talk box. Mixed: angle chase. Exit: say the bridge in one sentence.

Tuesday. Warm-up: vertical angles. Model: flow → paragraph. Attempt: student writes paragraph with reasons. Question: corresponding parts with a finger. Mixed: yesterday's flow. Exit: which step used a given?

Wednesday. Warm-up: sketch a rotated figure. Model: same proof, two-column option. Attempt: format swap. Autopsy: "SAS" with wrong included side. Mixed. Exit: name corresponding vertices in order.

Thursday. Warm-up: distance on a grid. Model: tiny coordinate proof that two segments are congruent. Attempt: student coordinate check. Mixed: synthetic + coordinate. Exit: one reason that used algebra.

Friday. Diagnostic wrong: formula-only area on a composite figure. Repair with reason + units. Out-of-home: ramp or bridge angle if safe. Done-enough checklist. Stop.

Adjust the focus skill if this week is similarity, parallel lines, or Pythagorean meaning — same skeleton, different bridge.

Practice that actually builds learning

Blocked, for a new move. The day you introduce know/need/bridge, only outlines, one figure type. The day you introduce format swap, only one proof rewritten two ways. Mixing too early makes the student hunt for "the look of a proof."

Mixed, for when to use it. Later the same week, an angle chase next to a congruence proof next to a coordinate check. Mixing is the practice of choosing: calculation, criterion, or coordinate bridge?

One incorrect example to diagnose. Formula card only; columns without a global argument; "SAS" with mismatched parts; two-column-only dogma; SOHCAHTOA with no right triangle. Hear the missing reason. The repair is a better question, then the figure, then — if needed — a named fact: "Corresponding parts in order," or "Formats are options; reasons are required." Silence in the face of a formula-without-reason is not kindness.

These are illustrations, not reported families. Classroom proof-progression craft (flow then paragraph; gradual release) is teaching practice literature, not a causal home trial.[45]

Named try-it: Know / need / bridge outline (in-home)

Time. Fifteen to twenty minutes. Then stop.

Materials. One clear figure; scrap paper; pencil.

Safety. Ordinary table. Curious tone.

The fun. Saying the bridge before any columns exist.

The skill. Deductive structure before format.

Write three headings: Know (givens). Need (goal). Bridge (criterion or key relationship). Fill them in speech first, then in writing. Only after the bridge is spoken may the student choose two-column, paragraph, or flow for the write-up.

Named try-it: Proof format swap (in-home)

Time. Fifteen to twenty minutes.

[45] Classroom proof-progression and flow-chart proving craft (e.g., Nirode practice pieces; Miyazaki and colleagues on flow-chart proving) = teaching craft, not WWC product ratings and not a U.S. homeschool RCT.

Materials. Same triangle proof; two blank pages.

Safety. No shaming about handwriting or "real" proof looks.

The fun. Watching the same logic wear different clothes.

The skill. Formats optional; reasons required.

Write the argument as a flow. Then rewrite as paragraph *or* two-column. Check: every arrow became a reason; no reason vanished. If the student insists only one format is real, swap once on purpose this week.

Named try-it: Bridge or building angles (out-of-home)

Time. Fifteen to twenty minutes outside; short write-up at home.

Materials. Baseline measure (tape or known stride); way to estimate or measure an angle to the top (clinometer app used as a *measuring aid after* the triangle is drawn, or a simple homemade clinometer); scrap.

Safety. Roads and sidewalks. No climbing for a "trig photo." No trespassing. Ordinary courtesy. Measurement error is part of the math talk — not a failure.

The fun. "How tall is that… with a triangle?"

The skill. Right-triangle trig edge; similar triangles / tan as opposite over adjacent; reason before calculator value.

Draw the right triangle. Name opposite and adjacent. Write the ratio. Compute. Discuss why a one-degree error in angle changes height. Hand full unit-circle trigonometry to Chapter 4.

Named try-it: Ramp as similar right triangle / slope (out-of-home)

Time. About ten minutes looking; short write-up at the table.

Materials. Eyes; optional tape measure; scrap.

Safety. No climbing stunts. Ramps and stairs you already use.

The fun. Rise/run and angle in one figure.

The skill. Reason before formula; connect Algebra I slope (Chapter 1) to geometry similarity.

Compare a shallow ramp and a steeper one — or one ramp and a drawn similar triangle. Ask what stays in proportion. Write slope as rise/run and as a tangent ratio once the right angle is marked. Cars and ramps motivate. The marked figure and the reason still happen.

Talk box

Opening question (locked): "What are we given? What must we show? What bridge gets us there?"

Follow-ups

1. Say the reason out loud before you write the format.
2. Which parts correspond? Show me on the figure with your finger.
3. Show me another way to see it — coordinate, transformation, or a different criterion.
4. Pick a format once the logic is clear.
5. You may change your mind. Try the argument in a second format.

How to wait

Ask. Count a slow three in your head. Look at the figure, not at the student's face, if the silence is hard. After they stop talking, wait again. If they are mid-reason, do not cut them off. Three seconds is a convention, not a sacrament.

What a stuck silence usually means

They reached for a formula card with no representation. Or they are filling columns without a global argument. Or the figure is rotated and corresponding parts are lost. Or wait-time after the question was zero. Or they think only two-column counts. Or they are guessing what you want. Next move: smaller figure, mark givens together, wait, point back at the diagram. "What's the bridge?" is a hint. Writing column two for them is not a hint; it is the grab. Done enough this week: a spoken triad. "Given these marks. Goal: congruent triangles. Bridge: ASA on these angles and included side."

Practice literature, not a franchise. Teachers who write about proof progressions often suggest building a flow in an open situation, then in a closed textbook-like situation, then rewriting into paragraph or two-column. Steal the craft: structure first, format second. Do not treat any single article as a causal rating of your kitchen table.

Reading a proof is practice too. Once a week, read a short finished proof together — any format — and ask: Which statement is a given? Which step used a definition? Where did the bridge appear? Proof comprehension is part of learning to prove. It is not a waste of a "doing" day.

Composite figures and units. When area or volume is in the week, require a reason the figure allows the formula (right angles? parallel bases? which solid?). Name units. A number with no unit and no reason is not done-enough measurement.

Connect back to Algebra I without reprinting it. Slope as rise/run already lived in Chapter 1. Here it becomes parallel and perpendicular reasons on a coordinate proof, or a tangent ratio on a ramp. Same-to-both-sides still holds when an equation appears inside a coordinate argument. If that equation habit is soft, repair it without calling the year a failure.

Connect forward without dumping Chapter 4. Right-triangle trig in this chapter is an edge: identify the right triangle, name opposite/adjacent/hypotenuse, write the ratio, compute. The unit circle, radians, and periodic function talk wait for precalculus. SOHCAHTOA as the entire trigonometry course is the residue to refuse.

For the student

This page is for you.

What are we given? What must we show? What bridge gets us there?

Geometry that means something starts with those three questions. A proof is not calligraphy in two columns. It is a path from what you know to what you must show, with a reason at each step.

Two-column, paragraph, and flow-chart are three ways to write the path. You may use any of them once the logic is clear. What you may not skip is the reason.

A tiny worked example

Given: In triangle ABC and triangle DEF, angle A is congruent to angle D, side AB is congruent to side DE, and angle B is congruent to angle E.

Goal: Triangle ABC is congruent to triangle DEF.

Bridge: ASA — two angles and the included side (careful: name the included side between the angles you marked).

Flow sketch: mark givens $\rightarrow$ identify included sides AB and DE $\rightarrow$ ASA $\rightarrow$ congruence $\rightarrow$ corresponding parts follow.

Paragraph version: "Angles A and D are congruent, sides AB and DE are congruent, and angles B and E are congruent, with AB and DE included between those angles. By ASA, the triangles are congruent."

Two-column version (sketch): each statement on the left gets a reason on the right — given; given; given; ASA. The rows are not decoration. They are the same bridge written as pairs.

Someone writes "SAS" while pointing at a non-included side, or fills six empty-looking rows with no bridge. That skipped the argument.

Another tiny picture — measurement with a reason

A composite shape is a rectangle with a right triangle attached. Someone multiplies "length times width" for the whole outline and stops. Ask what pieces are present. Find the rectangle area with a reason. Find the triangle area with a reason (right angle identified). Add. Name square units. The formula sheet was not wrong; it was incomplete until the figure was decomposed.

Two tries

1. On a figure your parent marks, say given, goal, and bridge out loud before writing anything. Then write a short proof in the format you choose.
2. Rewrite that same proof in a second format. Or mark corresponding parts on a rotated figure and explain why the criterion still applies.

Explain it back

Tell someone at the table why formats are options and reasons are required. Show a formula (area or Pythagorean) as a summary that still needs a reason the figure allows it.

Challenge

Someone always fills two columns and cannot say the bridge in one sentence. What question do you ask them? Someone says paragraph proofs "don't count." How do you show the same reasons in paragraph form?

You are allowed to struggle. You may mark the figure, draw a flow, use scrap paper. You write. You talk. If you get stuck, ask for a hint — not the finished proof. Then try again.

When you talk, a sentence about given, goal, and bridge is enough for today. You do not have to finish a publisher's entire Geometry book this week. Today you reason.

A picture a computer made of a perfect polyhedron is a scene. It is interesting. It is not the marked diagram on your scrap.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The student recites formulas or congruence abbreviations with no reasons, or cannot mark corresponding parts on a rotated figure.

The formula sheet is doing the work of the argument. Next move: know/need/bridge only, for several days — no full write-up until the bridge is spoken. Finger on the figure every time. Stay here if this is still the bottleneck. Long two-column sets will not hold on top of unmatched parts. A human who will wait with the figure, not a solver that dumps a proof, is a reasonable next step if matching parts is still the whole hour after a stretch of daily work.

2. The student fills two-column busily with no global argument, or refuses every format except one.

Format fetish arrived without logic. Next move: flow or outline only. Then one format swap this week on purpose. Ask "what's the bridge?" before any new

row. Go ahead once they can state given/goal/bridge and write a short proof with reasons in *any* accepted format. A tutor is useful if column-filling without bridges remains the default after a couple of weeks *and* the hour has become a fight.

3. The student freezes on every proof, or the silence after your question is a wall, or right-triangle trig is chanted with no triangle identified.

The wait was zero, or the figure is too crowded, or they are guessing what you want, or Algebra I distance/slope tools are still soft for a coordinate bridge. Next move: shorter sits, simpler figures, the opening triad, wait a slow three twice. If coordinate proofs are in the week, repair Chapter 1 slope and distance first without shame. If silence is hard, you look at the figure, not at them. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that writes the proof, is the release valve.

When to slow down. Formula-only answers. Unmatched corresponding parts. Columns with no bridge. Sits so long that marking the figure never starts. Those are brakes. "Not ready" because of age is the brake this book will not use. "Not ready" because reasons are not yet spoken is a real brake. A Geometry workbook cover is not a reason to skip the triad.

When to go ahead. The student can mark givens on a diagram; write a correct short proof in any accepted format with reasons; name the congruence or similarity criterion used; compute with Pythagorean or right-triangle trig when the figure warrants, with the right triangle identified; and refuse formula-only area on a composite figure without units or reason. A format swap has happened at least once. Short sits are ordinary. Then Algebra II function families (Chapter 3) and precalculus trig (Chapter 4) have diagram habits to lean on. Being "good at geometry" because a student memorized abbreviations is not a reason to skip bridges.

When to get a human tutor. You have run the outline, or the format swap, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear a formula without a reason. Outsourcing the hearing is the thing to avoid, not asking for help.

Correct a missing reason without crushing the attempt. "I hear SAS. Show me which parts — with your finger — and say why they match." Then look. Hearing a wrong answer, asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The student attempts first. You hold the figure and the triad. A tool may explain today's idea *to you* from this chapter, suggest an isomorphic proof figure you then vet (answer key held by you), write a short SCRIPT for *you* to say after the student has tried, offer a hint after an attempt, or help you diagnose work the student already produced. Crop to the paper. Do not upload the student's face. The student never sees the key.

Ages 15–16: parent co-holds; parent still in the room. Ages 17–18: more agency; same bans; still not a secret friend during the attempt. Unaided first. SCRIPT after the try. Labelled GeoGebra or a geometry app after the attempt can explore a construction — eyes open, not a substitute for marked reasons on paper.

Leave these out of the hour: a chatbot as the only partner during the attempt; a tool that completes the proof worksheet; photo-to-key on a diagram; a detector score; a certificate of Geometry fluency; a cloud agent as a math partner. Photo-to-answer is the ban.

A language model will happily dump a two-column proof. Treat every model-supplied proof as untrusted until the student has stated given, goal, and bridge and written reasons they can defend. The same high-school math field experiment named in the math hour still applies: an unguarded chatbot can make practice look better and leave unaided performance worse.[46] This is a useful study, not a promise that every home will see the same result. The model may prepare the adult and the next figure; it may not do the student's proof during the attempt.

A marked diagram, a flow on scrap, a homemade clinometer used safely, and a finished proof you read together are tools too. Use them, then fade them.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 10 Geometry workbook" is a publisher's scope, not a legal grade, and not a transcript line. The record, when you

[46] Bastani et al., *PNAS* 122, no. 26 (2025): e2422633122 — ON-AGE high-school math field experiment; see math-hour box and Chapter 1 notes. Not a U.S. homeschool RCT.

need one, is a dated notebook, the task named, plus an exit ticket, plus one diagnostic item. The title a stranger can read is **Geometry** — only if the year's work was that course. It is not Emerging Adults Math I. It is not a brand name as the credit.

Checklist before moving on

- You can hear a formula without a reason, and you can ask what must be true and how they know.
- You can hear column-filling with no bridge, and you can ask the locked triad.
- The student marks givens on a diagram before writing a proof.
- The student writes a correct short proof in *any* accepted format (two-column, paragraph, or flow) with reasons.
- The student names the congruence or similarity criterion used and can show corresponding parts on a rotated figure.
- At least one format swap has shown that formats are options and reasons are required.
- When the week includes measurement: Pythagorean or right-triangle trig appears with the right triangle identified and units where they belong.
- At least one in-home try-it (outline or format swap) and one out-of-home try-it (bridge angle or ramp slope) landed on paper.
- Short sits are ordinary. A spoken given/goal/bridge is enough for an exit. You did not require college geometry.
- You can hear formula-sheet-only, SAS-without-parts, two-column dogma, and rotated-figure panic, and you can ask a good question, without taking the pencil.

If most of that list is true, go on to Algebra II function families, even if the birthday says otherwise. If the birthday says "Precalculus" and reasons are still absent, stay. Chapter 3 asks for families without a regression button. Chapter 4 asks where an angle sits on the unit circle. Both need habits of saying why.

A path through high-school mathematics is the promise. A diploma is not. A percentile is not. An AP score is not the only path that counts.

Chapter 3

Algebra II: function families

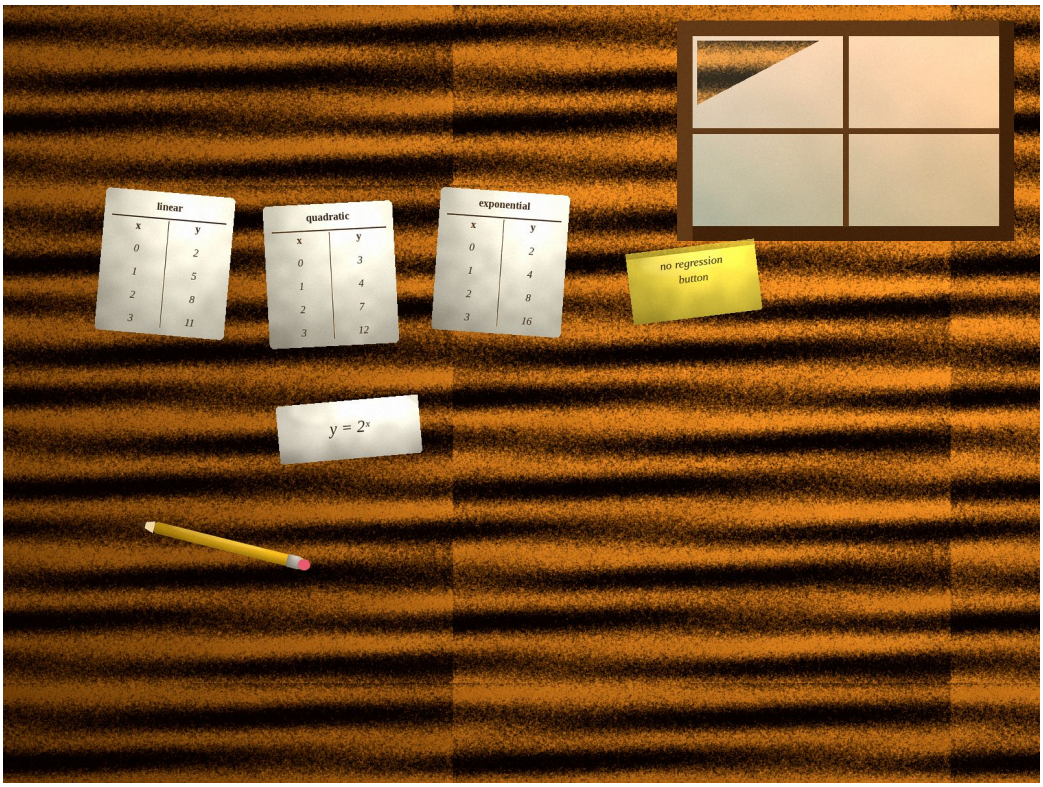

Figure 6: A kitchen-table still-life: three cards showing a linear table, a quadratic table, and an exponential table; a sticky note reading “no regression button”; scrap with a transformed parent sketch. No people. No logos.

Why this matters

Is this table linear, quadratic, or exponential — how do you know without a regression button?

That question is the whole chapter in one breath. Algebra II in this book is not a second round of letter-chasing. It is learning to see function families — polynomial, rational, exponential, logarithmic — as structures you can recognize from a table, a graph, and a situation, before you chase symbols. When the family is clear, the equation becomes a compressed description of something you already understand.

Many adults remember Algebra II as "factor harder polynomials, then move letters until the log properties appear." Procedures matter. They are a weak first story. The first story is: sort the family; name the parent; say what a transformation did; match a growth or decay context to the structure; then solve with eyes open — including extraneous-root checks on rationals.

Structure before letter-chasing is the same Star grain that carried Chapter 1, still live here: notice what different representations show; choose strategies on purpose once some competence exists.[47] Function literature at the conceptual grain presses covariation and input-output meaning over "function means write `f(x) =` and stop."[48] Those sources are a useful guide, not a promise that every home will see the same result.

What this idea unlocks is precalculus and calculus readiness without birthday theater. Unit-circle trigonometry needs function fluency. Limits and rates need families you can already tell apart. Statistics later will ask what an average hides; rate language starts here. If every table is "hit regression," every new context is a button hunt. If differences and ratios are read by hand first, the student can change their mind when the graph disagrees with a guess.

Why teach this *now*, at fifteen to eighteen? Because Algebra II is a common high-school pathway, not a universal finish line by seventeenth birthday — federal

[47] Jon R. Star et al., *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students* (NCEE 2015-4010), April 2015. Recommendations on solved problems, structure, and alternative strategies remain on-age for Algebra II. Access date for URLs in these notes: 5 September 2026.

[48] Conceptual function grain associated with Carlson, Oehrtman, Thompson and colleagues: covariation and input-output meaning over formula-only definitions — teaching implication for this chapter's family sorts and transformation talk.

parent-survey rows for grades 9–12 that year showed Algebra II as a frequent offering beside unfinished arithmetic and Algebra I, which is a reminder to place by skill.[49] Because what is developmentally appropriate depends on prior opportunities to learn.[50] Because ages 15–16 may begin early Algebra II edges (simple exponentials, transformed parents, rational repair) while finishing Algebra I/Geometry meaning; ages 17–18 typically complete function-family fluency — if the prerequisites are present.

This book is not a reprint of *Math for Young Minds* early functions. One pointer: linear thinking continues. Then we teach families at Algebra II grain. It is not a reprint of *Mathematics for Homeschooling*. One pointer, then teach this week's object. It is not a dump of full AP Precalculus into this chapter — Chapter 4 owns unit circle and parent-function depth toward calculus.

You do not need to be a mathematician. You do need to hear regression-button family ID as a missing-structure move, not a cute slip; to hear every growth story treated as linear; and to ask how they know the family without the button, without taking the pencil.

This week you can learn to hear letter-chasing with no family, skipped extraneous-root checks, and "just use the formula" before a table. Today the student can sort three tables into families and write one transformed parent function with a sentence about what changed.

For the parent: understand it yourself

Many adults feel rusty on Algebra II they last saw as a student. That is ordinary. A diet of symbol drills without family talk trains letter-chasing. Five minutes of this section, then the warm-up at the end, is enough for tomorrow. The student still writes the structure.

Everyday picture. A car-loan amortization row: payment is steady; interest portion changes. Something linear (the payment) sits beside something that compounds (the balance's interest). Or a sports box score: points per game as a rate rule; season total as accumulation of that rule — without turning the evening into

[49] NCES PFI 2016 grades 9–12 subject rows *that year* (Algebra II among common offerings) — place by skill; not a kitchen promise about your student. See research brief for copied percents.

[50] National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 15 — place by skill; prior opportunities to learn matter more than age-stage slogans.

a statistics course (Chapter 6 owns inference talk). Or a paycheck stub percent: withholding as a percent of gross is a multiplicative structure, not a keyword.

Kitchen and cars motivate. The table, the graph, and the family name still happen. A car-loan row is a math object. It is not a sermon about debt culture. Sports stats are rates. Paycheck percent is structure. Pharmacy dosage-as-rate, if you ever use it as a composite illustration, is **not medical advice**.

Precise picture. A **function** pairs inputs with outputs so that each input has exactly one output. A **family** is a set of functions that share a structure:

- **Linear:** constant first differences on equal input steps; constant rate; parent often `y = x` or `y = mx + b`.
- **Quadratic (degree-2 polynomial):** roughly constant second differences; parent `y = x^2`; Chapter 1 continuity.
- **Higher polynomials:** degree shows in how differences eventually stabilize at higher order; end behavior follows leading term — teach with graphs and tables, not only vocabulary lists.
- **Exponential:** constant *ratios* (multiplicative growth or decay) on equal input steps; parent `y = a·b^x` with `b > 0`, `b` $\neq$ `1`; growth if `b > 1`, decay if `0 < b < 1`.
- **Logarithmic:** inverse relationship to exponential; log asks "what exponent?"; parent `y = log_b x`.
- **Rational:** quotient of polynomials; watch domain (denominator $\neq 0$); solve and **check for extraneous roots** introduced by clearing denominators.

Transformations. From a parent, `y = a·f(x − h) + k` (and reflections) shift, stretch, and flip. The student should say in a sentence what changed: "shifted right 2, stretched by 3, up 1," while pointing at a graph — not only rewrite letters.

Strategy choice, still live. Solve an exponential by rewriting with the same base *or* by logs once logs are taught; compare. Solve a quadratic factor *or* formula *or* graph. Flexibility after competence — same Star note as Chapter 1.[51]

Study a solved table sort before racing blanks. Linear table `x = 0,1,2,3` $\rightarrow$ `y = 2,5,8,11`: first differences `3,3,3` — constant rate. Quadratic `y = 1,2,5,10`:

[51] Jon R. Star et al., *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students* (NCEE 2015-4010), April 2015. Recommendations on solved problems, structure, and alternative strategies remain on-age for Algebra II. Access date for URLs in these notes: 5 September 2026.

first differences `1,3,5`; second `2,2` — quadratic family. Exponential `y = 3,6,12,24`: ratios `2,2,2` — exponential family. No regression button required.

Here is a correct exponential solve with structure, once same-base thinking is in place:

`2^x = 32`

Rewrite 32 as `2^5`. Then `2^x = 2^5`, so `x = 5`. Check: `2^5 = 32`. True.

Later, with logs taught as inverse talk: `x = log_2 32`, same meaning. The log line is not a different mathematics; it is a name for the exponent question.

Here is an incorrect twin you should hear: someone fits a linear regression to the exponential table because the calculator made an `r` look comforting, then cannot explain why doubling happened. The miss is family-blind procedure.

Another incorrect twin: someone solves a rational equation, gets `x = 3`, never substitutes, and does not notice that `x = 3` zeros a denominator. The miss is skipped checking.

Wrong answers you should be able to hear

1. *Hitting the regression button before looking at differences or ratios.* Ask the locked opening. Cover the calculator. Hand table first.
2. *Letter-chasing with no family — rearranging symbols with no graph, table, or situation.* Ask: "What is the parent function, and what transformations happened?"
3. *Treating every growth story as linear — "it goes up, so add."* Ask for ratios on equal steps. Constant add vs constant multiply.
4. *Skipping the extraneous-root check on a rational equation.* Clearing denominators can introduce solutions that make a denominator zero. Substitute back into the original.
5. *Saying what a transformation did only as letter soup, with no sentence and no graph finger-point.* Ask them to show the shift on the graph and say it in plain words.

A sixth you will also hear: "I'm seventeen, so Algebra II is done," while families still blur. Place by skill.[52]

[52] National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 15 — place

Five-minute parent warm-up

Do this before the lesson, on a scrap of paper, no student in the room.

Minute 1. Build three tiny tables: linear, quadratic, exponential. Compute differences or ratios. Name each family out loud.

Minute 2. Sketch `y = 2^x` and `y = x^2` for a few positive `x`. Say one sentence about how the exponential pulls away.

Minute 3. Rewrite `y = (x − 2)[^41] + 1` as "parent `x^2`, right 2, up 1." Point in the air where the vertex moved.

Minute 4. Invent a wrong rational solve that forgets to check a value that zeros a denominator. Mark the miss.

Minute 5. Write the sentences you will actually say: "Is this table linear, quadratic, or exponential — how do you know without a regression button?" Under it: "What is the parent, and what changed? Check extraneous roots." Put the pencil down. Those sentences are the lesson.

If you can do those five minutes, you are ready to sit down. The student writes the structure. You hear.

How to teach it this week

A good math hour this week has a shape. The math-hour chapter in the front of this book is the full version. Here is the shape scaled to this idea. Typical sit: forty-five to seventy-five minutes; stretch when a finance table or a rational repair needs it.

Warm-up (5–10 minutes, unaided). Evaluate a known exponential or a transformed linear parent at two inputs. One tiny "which family?" table. Paper. No regression. No photo-to-key.

Short model (8–12 minutes). Table sort: linear / quadratic / exponential side by side. Narrate differences vs ratios. Optional: one transformed parent with a graph sketch. Incorrect twin: regression-first or linear-fit on exponential data. You talk for a few minutes. Then you stop.

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by skill; prior opportunities to learn matter more than age-stage slogans.

Student attempt (15–30 minutes). Sort three new tables with written justifications; or write a transformed parent and a growth-context sentence; or solve a routine exponential/log/rational item *after* the family is named. You wait.

One good question, then wait. Use the locked opening. A slow three. Look at the table, not at the student's face, if the silence is hard. Stahl's think-time is a classroom convention; Rowe's windows were science class.[53] Use the pause.

Mixed practice (10–15 minutes). A rational repair next to an exponential evaluate next to a quadratic from Chapter 1. Mixing is how the student learns *when* to use ratios vs second differences vs both-sides algebra.

Exit ticket (3–5 minutes). Family ID without regression. One-sentence transformation or context. Optional check of an extraneous root. Done-enough is right, or wrong-with-a-reason we can use tomorrow.

Exact wording you can say

"Is this table linear, quadratic, or exponential — how do you know without a regression button?"

"What is the parent function, and what transformations happened?"

"Show graph and equation — same relationship?"

"Does this growth context match the family?"

"Substitute back — any extraneous roots?"

"You may change your mind."

When they reach for regression first:

"Differences or ratios on the table first. Then we can check with a tool."

When they treat growth as linear:

"Are we adding the same amount each step, or multiplying by the same factor?"

When you are about to take over:

"Your pencil. I'll wait."

Age-band moves: 15–16 / 17–18

[53] Rowe (1972) science wait-time; Stahl (1994) think-time ~3 seconds — classroom-origin conventions, not a homeschool algebra RCT.

15–16. Early Algebra II edges as skill allows: function-family sort (linear / quadratic / exponential) from tables and graphs; simple exponential evaluate and growth/decay sentences; transformed parents for linear and quadratic (and simple absolute value if in-course); rational expressions repair with domain awareness; log as "inverse talk" once exponentials are solid — not a vocabulary dump first. If Algebra I linear/quadratic structure from Chapter 1 is soft, stay there without apology. Geometry reasons (Chapter 2) can run in parallel weeks; keep them as separate sits.

17–18. Complete Algebra II function-family fluency: polynomials beyond quadratic with end-behavior talk; rational equations with extraneous-root discipline; exponential and logarithmic equations with structure (same base, inverse, properties as rewrite tools — not fifty identities at once); richer transformations; finance or growth contexts that land on structure. If families still blur, repair sorts before precalculus. Turning eighteen does not finish Algebra II by candlelight. Ready students may edge toward Chapter 4; birthday alone does not.

A fifteen-year-old who can sort families and write a transformed parent is not "too young" for this spine. A seventeen-year-old who cannot is not "behind on birthday calculus" — they are behind on function meaning.

First try-it for the student

On scrap paper:

Three short tables. Ask: "Is this table linear, quadratic, or exponential — how do you know without a regression button?"

Wait. If they guess from the shape of the numbers without computing, ask for differences or ratios written out. If they ask for the calculator's regression, cover it and point at the table.

Later the same week:

Write `y = 2(x + 1)[^41] − 3` from parent `y = x^2` in a sentence: what stretched, what shifted. Sketch. Or solve a rational equation and check candidates in the original.

How to fade help. First sitting: you compute differences/ratios, they name the family. Second: they compute, you wait, you hint ("ratio or second difference?"). Third: they sort without your marks. Fourth: a growth context that must match the family in a sentence. Logs and harder rationals fade in after exponential and

domain habits exist.

When to stop talking. When you hear yourself lecturing *Laws of Logarithms* as a list. When the sit becomes a regression demo. When the student is mid-sort and you stack questions. One good question beats five. Stop while they still have a table left in them.

This week's plan (honest "lesson plan"). One focus skill — for example, exponential vs linear/quadratic from tables; or log as inverse talk; or rational extraneous-root check. Four or five sessions. One in-home try-it and one out-of-home try-it below. Friday: one diagnostic wrong (regression-first or skipped check) plus done-enough. Not one hundred eighty worksheets. Not a percentile.

Sample week (family ID $\rightarrow$ transformed parent $\rightarrow$ context).

Monday. Warm-up: evaluate `2^x`. Model: three-table sort. Attempt: justify with differences/ratios. Talk box. Mixed: Chapter 1 quadratic. Exit: family ID.

Tuesday. Warm-up: one linear table. Model: `y = a·b^x` meaning of `a` and `b`. Attempt: write a growth sentence that matches. Mixed. Exit: growth or decay?

Wednesday. Warm-up: sketch `y = x^2`. Model: transformed parent. Attempt: write and sketch one transform; say what changed. Autopsy: letter soup with no sentence. Exit: point to the vertex shift.

Thursday. Warm-up: domain of a simple rational. Model: clear denominators + check. Attempt: solve and reject extraneous. Mixed: exponential evaluate. Exit: which candidate failed and why?

Friday. Diagnostic: regression-button reflex on an exponential table. Repair with ratios. In-home APR rows or out-of-home sports/paycheck percent. Done-enough checklist.

Practice that actually builds learning

Blocked, for a new move. The day you introduce constant ratios, only exponential vs linear tables, friendly numbers. The day you introduce extraneous-root checks, only that repair. Mixing too early makes the student hunt for "the button."

Mixed, for when to use it. Later the same week, a family sort next to a rational check next to a transformation sentence. Mixing is the practice of choosing:

differences, ratios, domain check, or parent talk?

One incorrect example to diagnose. Regression-button family ID; letter-chasing with no family; every growth story treated as linear; skipped extraneous-root check; transformation as letter soup. Hear the missing structure. The repair is a better question, then the table or graph, then — if needed — a named fact: "Constant ratios mean exponential family," or "Check candidates in the original rational equation." Silence in the face of a regression reflex is not kindness.

Contexts help meaning. They do not automatically raise scores — keep ordinary stakes and land on structure.[54] These are illustrations, not reported families.

Named try-it: Function-family sort cards (in-home)

Time. Fifteen to twenty minutes. Then stop.

Materials. Ten cards: tables and graphs (hand-drawn is fine); scrap for differences/ratios.

Safety. Ordinary table. No race to a calculator.

The fun. Sorting and defending.

The skill. Family ID without a regression button.

Sort into linear / quadratic / exponential (add "other/not sure" if needed). Justify each with first differences, second differences, or ratios. Disagreement is useful: recompute together. Desmos or a graphing calculator may check *after* the hand sort — labelled, eyes open.

Named try-it: Car-loan APR three-row amortization (in-home)

Time. Fifteen to twenty minutes.

Materials. Principal, APR, term as a composite example; formula or spreadsheet the parent checks; scrap.

[54] Contexts for meaning are not a claim that every "real world" programme raises scores; named secondary trials vary. Keep ordinary stakes; land on structure.

Safety. Math object only — not a politics seminar, not a debt sermon, not a product pitch. Use rounded educational numbers.

The fun. Seeing interest portion change while payment stays steady.

The skill. Exponential/compound structure beside a linear payment; read three rows; explain month-1 interest vs a later month in one sentence each.

Build or read three rows. Ask what is constant and what is changing. Connect to family talk: payment as repeated subtraction from a balance that also grows by a rate — structure, not fear.

Named try-it: Percent change with a delayed quote (in-home)

Time. Ten to fifteen minutes.

Materials. Historical printout or delayed educational price/index quote — not a live "buy this" tip.

Safety. No stock tip. No live controversy. Absolute change vs percent change as math.

The fun. "Is a five-dollar move big?"

The skill. Percent change with meaning; multiplicative thinking that supports exponential talk.

Compute absolute difference and percent change. Ask which measure matches the question being asked. Land on paper. Hand deeper inference to Chapter 6.

Named try-it: Sports rate as function talk (out-of-home)

Time. During a game or while reading a box score; five to ten minutes of talk; short write-up at home.

Materials. Box score or live observation; scrap later.

Safety. Ordinary courtesy. Sports rates are rates — not identity fights.

The fun. "Points per game as a rule."

The skill. Rate language; total vs rate; what an average can hide (depth to Chapter 6).

Write a simple rule: points $\approx$ rate $\times$ games. Ask what the average hides if one game was an outlier. Function talk, not a full stats course.

Named try-it: Paycheck or store percent (out-of-home $\rightarrow$ table)

Time. Ordinary errand or pay-stub look; ten to fifteen minutes at home.

Materials. Receipt or stub the family already understands; scrap.

Safety. Ordinary aisle courtesy. No public quiz of a cashier. Paycheck percent is a math object.

The fun. "Withholding as a structure."

The skill. Percent as multiplicative structure; land on paper; optional link to exponential "apply the same factor again" talk without forcing a full model.

Write gross $\rightarrow$ percent $\rightarrow$ net as steps. Ask what stays in proportion if gross changes. If percent meaning is still soft from Foundations, repair that first without shame.

Talk box

Opening question (locked): "Is this table linear, quadratic, or exponential — how do you know without a regression button?"

Follow-ups

1. What is the parent function, and what transformations happened?
2. Show graph and equation — same relationship?
3. Does a growth context match the family?
4. If this is rational, which candidates need an extraneous-root check?
5. You may change your mind. Try differences and ratios another way.

How to wait

Ask. Count a slow three in your head. Look at the table or graph, not at the student's face, if the silence is hard. After they stop talking, wait again. If they are mid-reason, do not cut them off. Three seconds is a convention, not a sacrament.

What a stuck silence usually means

They reached for the regression button. Or they are letter-chasing with no family. Or wait-time after the question was zero. Or they are hunting a keyword on a word equation. Or they treat all growth as linear. Or the numbers or the structure are too new. Or they are guessing what you want. Next move: smaller tables, hand differences, wait, point back at ratios. "Show me the ratios" is a hint. "It's exponential because I said so" said by you is not a hint; it is the grab. Done enough this week: a spoken justification. "Ratios stay at 2 — exponential. Check: next term should double."

Polynomials beyond the friendly quadratic. Once linear and quadratic sorts are solid, bring a cubic table or graph and ask what the differences do at the next order — or simply compare end behavior on a sketch: which way do the arms go, and what does the leading term suggest? You do not need a full coefficient lecture on day one. You need eyes on degree as a family clue.

Logs as inverse talk, not a laundry list. When `y = b^x` is comfortable, ask what input produces a given output. That question *is* the logarithm. Properties of logs arrive as rewrite tools for equations you already understand ("same base," "undo the exponential"), not as fifty lines to memorize before any meaning. If the student can solve `2^x = 32` by inspection and then rewrite with logs once taught, structure is winning.

Inequalities and domain as family cousins. Exponential and rational work quickly raise "for which x is this defined?" and "when is this positive?" Keep same-to-both-sides alive from Chapter 1 when solving inequalities, including the flip when multiplying by a negative, with a reason. Domain holes on rationals are reasons, not decorations.

What this chapter is not. It is not Chapter 4's unit circle. It is not Chapter 5's limits. It is not a vendor ranking of Algebra II programmes — fit lives in the resources chapter. It is not a requirement that every seventeen-year-old finish the entire publisher scope before a birthday.

For the student

This page is for you.

Is this table linear, quadratic, or exponential — how do you know without a regression button?

Look at equal steps of the input. If the output changes by adding (or subtracting) the same amount each time, think linear. If the *change in the change* settles into a pattern, think quadratic. If the output multiplies by the same factor each step, think exponential. The calculator's regression can wait. Your eyes and a scrap column of differences or ratios come first.

A function takes an input and gives one output. Families share a shape of change. Parent functions are the simple versions. Transformations slide, stretch, and flip those parents. Your job is to say what changed in plain words while pointing at a graph.

A tiny worked example

Tables with `x = 0, 1, 2, 3`:

A: `y = 4, 7, 10, 13` — first differences `3, 3, 3` → linear.

B: `y = 0, 1, 4, 9` — first differences `1, 3, 5`; second `2, 2` → quadratic.

C: `y = 5, 10, 20, 40` — ratios `2, 2, 2` → exponential.

Someone runs linear regression on C because the points "go up" and then cannot explain the doubling. That skipped the family.

Transformed parent

Parent: `y = x^2`.

New: `y = 2(x − 1)[^41] + 3`.

Sentence: "Stretched by 2, shifted right 1, up 3." Vertex moves from `(0, 0)` toward `(1, 3)`. If you only move letters and cannot say that sentence, slow down and sketch.

Rational check

If you clear denominators to solve and get candidate `x = 2`, but `x = 2` makes a denominator zero in the original, reject it. Checking is part of solving.

Exponential with meaning

`2^x = 32` → rewrite as `2^x = 2^5` → `x = 5`. Or, once logs are in the room: `x = log_2 32`. Same question: what exponent on 2 gives 32? Someone "moves the 2" as if it were Chapter 1 letter-moving and writes nonsense. Ask them which family they are in first.

Two tries

1. Sort three tables your parent gives you. Write differences or ratios. Name each family.
2. Write one transformed parent from `y = 2^x` or `y = x^2` and say what changed. Or solve a rational equation and show the extraneous-root check.

Explain it back

Tell someone at the table how you know exponential without a regression button. Show a growth story that is *not* linear.

Challenge

Someone always hits regression first. What question do you ask them? Someone treats "it increases" as enough to call a table linear. How do you show constant ratios instead?

You are allowed to struggle. You may use cards, scrap columns, sketches. You write. You talk. If you get stuck, ask for a hint — not the finished equation. Then try again.

When you talk, a sentence about differences, ratios, or a transformation is enough for today. You do not have to finish Algebra II because of a birthday. Today you sort, transform, and check.

A picture a computer made of a perfect exponential curve is a scene. It is interesting. It is not the ratio column on your scrap.

If it isn't clicking

Three diagnostics. Each one has a next move. None of them is a verdict on talent, and none of them is a reason to wait for a birthday.

1. The student hits regression (or guesses "curved") before differences or ratios, or treats all growth as linear.

Procedure or vibe is doing the work of structure. Next move: hand tables only for several days — no regression key. Sort cards with written justifications. Stay here if this is still the bottleneck. Logarithm worksheets will not hold on top of family blur. A human who will wait with the table, not a solver that dumps a fit, is a reasonable next step if the button remains the whole hour after a stretch of daily sorts.

2. The student letter-chases transformations or equations with no graph sentence, or skips extraneous-root checks on rationals.

Symbols arrived without meaning or without domain discipline. Next move: transformation sentences with finger on graph only; rational solves that end with substitution every time. Go ahead once they can sort three tables, write one transformed parent in words, and reject one extraneous root on purpose. A tutor is useful if letter soup without graphs remains the default after a couple of weeks *and* the hour has become a fight.

3. The student freezes on every new family, or silence after your question is a wall, or Algebra I linear/quadratic structure from Chapter 1 is still soft.

The wait was zero, or Foundations/Algebra I gaps are carrying the hour, or they are guessing what you want. Next move: shorter sits, Chapter 1 difference tests first without shame, the locked opening, wait a slow three twice. If exponential ratios are new, compare only linear vs exponential for a week. If silence is hard, you look at the table, not at them. If the hour has become a fight after a stretch of daily short sits, a human who will wait, not a chatbot that fills the blanks, is the release valve.

When to slow down. Regression-first every time. Growth always called linear. Checks skipped. Sits so long that a sort never starts. Those are brakes. "Not ready" because of age is the brake this book will not use. "Not ready" because families still blur is a real brake. An Algebra II workbook cover is not a reason to skip the table.

When to go ahead. The student distinguishes linear / quadratic / exponential from a table without regression; solves a routine exponential or log equation taught in-course with structure; graphs or describes a transformed parent in a sentence; handles a rational equation with an extraneous-root check; explains a finance or

growth context sentence without keyword theater. Short sits are ordinary. Then precalculus and trigonometry (Chapter 4) have function fluency to lean on. Being "good at Algebra II" because a student is fast at moving letters is not a reason to skip families.

When to get a human tutor. You have run the sorts, or the checks, or the wait after the question, for a stretch of daily sittings, and the same diagnosis is still the one in the room, and the hour has become a fight. A tutor is a release valve, not a failure of the sitting. Keep yourself as the person who can still hear regression-first as a missing-structure move. Outsourcing the hearing is the thing to avoid, not asking for help.

Correct a wrong family call without crushing the attempt. "I hear linear. Show me the first differences — and the ratios." Then look. Hearing a wrong answer, asking a better question, then naming a better move if needed, is teaching.

Tools, including AI

Optional helpers for you, the adult. The full rules live in the math-hour box in the front of this book. This chapter is a pointer, not a second policy paper.

The student attempts first. You hold the table and the question. A tool may explain today's idea *to you* from this chapter, suggest isomorphic tables you then vet (answer key held by you), write a short SCRIPT for *you* to say after the student has tried, offer a hint after an attempt, or help you diagnose work the student already produced. Crop to the paper. Do not upload the student's face. The student never sees the key.

Ages 15–16: parent co-holds; parent still in the room. Ages 17–18: more agency; same bans; still not a secret friend during the attempt. Unaided first. SCRIPT after the try. Labelled Desmos, graphing calculator, or spreadsheet after the attempt can check a graph or an amortization row — eyes open, not a substitute for hand ratios. CAS tools, if used on calculator-allowed practice, stay labelled and still are not a chatbot finishing the free-response.

Leave these out of the hour: a chatbot as the only partner during the attempt; a tool that completes the worksheet; photo-to-key; a detector score; a certificate of Algebra II fluency; a cloud agent as a math partner. Photo-to-answer is the ban.

A language model will happily dump a regression equation or a log property list.

Treat every model-supplied solution as untrusted until the student has named the family and checked. The same high-school math field experiment named in the math hour still applies: an unguarded chatbot can make practice look better and leave unaided performance worse.[55] This is a useful study, not a promise that every home will see the same result. The model may prepare the adult and the next table; it may not do the student's problem during the attempt.

Sort cards, a hand ratio column, a parent-checked amortization snippet, and a wrong regression-first autopsy you wrote yourself are tools too. Use them, then fade them.

What "done enough" looks like

Placement is by skill, not birthday. A "grade 11 Algebra II workbook" is a publisher's scope, not a legal grade, and not a transcript line. The record, when you need one, is a dated notebook, the task named, plus an exit ticket, plus one diagnostic item. The title a stranger can read is **Algebra II** — only if the year's work was that course. It is not Emerging Adults Math I. It is not a brand name as the credit.

Checklist before moving on

- You can hear regression-button family ID, and you can ask for differences or ratios on the table.
- You can hear letter-chasing with no family, and you can ask for the parent and the transformation sentence.
- The student distinguishes linear / quadratic / exponential from a table without a regression button.
- The student writes or graphs a transformed parent and says what changed in plain words.
- The student solves a routine exponential or logarithmic equation taught in-course with structure visible (not only a memorized chant).
- The student handles a rational equation with an extraneous-root check by substituting into the original.
- A growth, finance, sports-rate, or percent context has landed on structure — motivated, not replaced the table.

[55] Bastani et al., *PNAS* 122, no. 26 (2025): e2422633122 — ON-AGE high-school math field experiment; see math-hour box. Not a U.S. homeschool RCT.

- At least one in-home try-it (family sort or APR rows) and one out-of-home try-it (sports rate or paycheck/store percent) landed on paper.
- Short sits are ordinary. A spoken family justification is enough for an exit. You did not require full AP Precalculus as proof of Algebra II.
- You can hear linear-for-all-growth, skipped extraneous checks, transformation letter soup, and "birthday finished Algebra II," and you can ask a good question, without taking the pencil.

If most of that list is true, go on to precalculus and trigonometry, even if the birthday says otherwise. If the birthday says "Calculus" and families still blur, stay or return to Chapter 1 structure first. Chapter 4 asks where an angle sits on the unit circle and why the sign is what it is. It needs function fluency underneath. Calculus (Chapter 5) is for ready students — not every eighteenth birthday.

A path through high-school mathematics is the promise. A diploma is not. A percentile is not. An AP score is not the only path that counts.

Chapter 4 — Precalculus and trigonometry

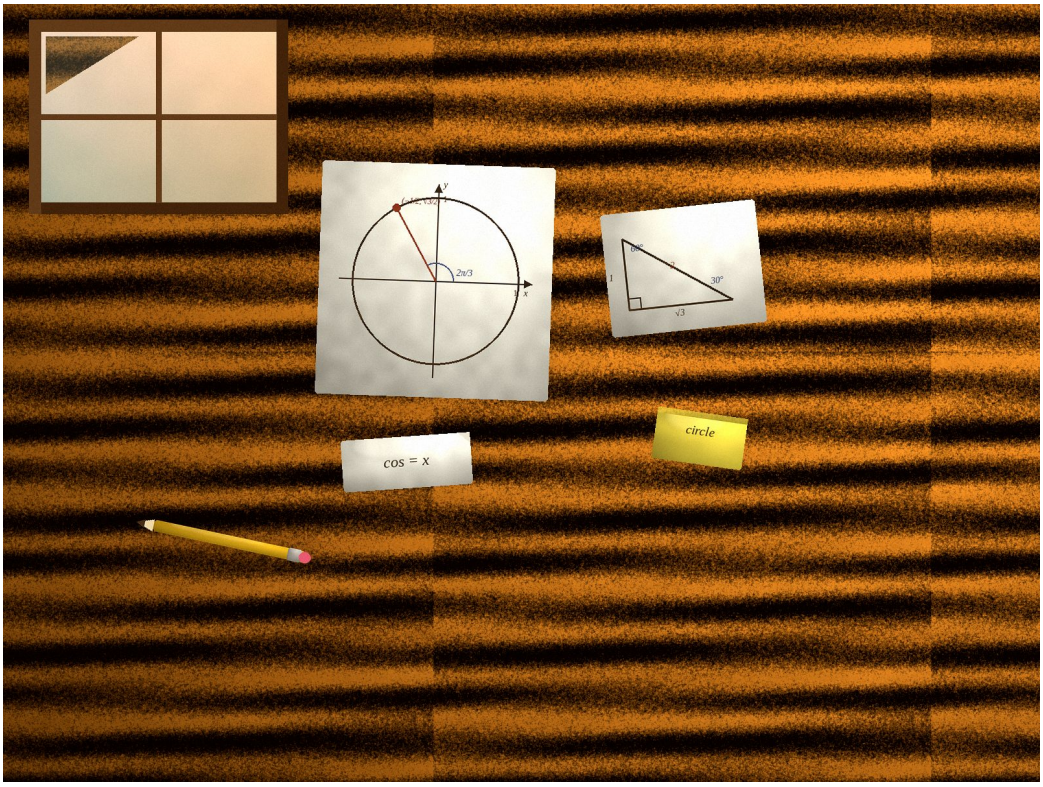

Figure 7: A kitchen-table still-life: unit circle on scrap paper with special angles marked; a tiny sine sketch with amplitude and period labelled; a protractor. No people. No logos.

Why this matters

Trigonometry that never leaves a right triangle is a useful tool. It is not the whole course.

When your student can place an angle on the unit circle, name the coordinates, and say *why* the sign is what it is, they own sine and cosine as functions — not as three letters on a triangle. That ownership unlocks special-angle values without a calculator, graphs with amplitude and period that mean something, and the kind of covariation talk that ready students will need when calculus begins.

Right-triangle trig and SOHCAHTOA can begin at 15–16, often inside geometry. Unit circle, radian measure, and function-level trig typically sit at 17–18 inside precalculus. Place by skill, not birthday. A fifteen-year-old who already thinks in coordinates may start the circle early with you beside them. An eighteen-year-old who still treats every trig problem as "opposite over hypotenuse" needs the circle before another identity list.

This chapter refuses two traps. The first is identity memorization with no circle underneath — fifty formulas week one, zero pictures. The second is "SOHCAH-TOA" as the entire trig course. Map bearings and bridge angles will motivate. The circle and the written values still have to appear.

Why struggle through the circle when the calculator will spit out a decimal? Because the decimal does not tell you *why* cosine is negative in the second quadrant, does not give you exact values for later algebra, and does not transfer to the sine graph you must sketch by hand. The circle is the meaning. The calculator is a check after the meaning.

Parent functions from Algebra II still live here. You transform y = x the same way you transform y = x^2 — shift, stretch, reflect — once you know what the parent looks like. Watching two quantities change together ("as θ grows, what is y doing?") is the bridge toward calculus. You are not teaching calculus yet. You are teaching the habit.

This work lives in the home and outside it. The dinner-table circle and the scrap-paper reference angle are not lesser than the bridge or the trail map. The bridge and the map are not a substitute for the circle. Both kinds of try-it belong in the week.

CCSS-M Geometry, Functions, and Number categories can sit in the background

as a map of topics — not as a statute in your kitchen, and not as a script you must recite. There is no dedicated IES high-school trigonometry practice guide this book leans on, so the teaching claim stays modest and practical: representation before procedure; circle before identity laundry list; application with a diagram; place by skill.

Parent learns this week: hear calculator-first for $2\pi/3$, and ask for the point on the circle. Student tries today: place three special angles with coordinates and evaluate sine and cosine without a calculator.

For the parent: understand it yourself

You do not need to be a mathematician. You do need to hear SOHCAHTOA-only, calculator-first special angles, and a sine sketch with no amplitude meaning — and to ask a good question without taking the pencil.

One everyday picture

Imagine a clock face remade as a circle of radius 1. The center is the origin. The rightmost tip of the horizontal diameter is the point (1,0). An angle is measured from that tip, counterclockwise for positive angles. Wherever the ray lands on the circle, that landing point has coordinates (x,y). By definition, $\theta = x$ and $\theta = y$.

That is the unit circle. Cosine is the horizontal coordinate. Sine is the vertical coordinate. Tangent is y/x when $x \neq 0$. The signs follow the quadrant: in the second quadrant x is negative and y is positive, so cosine is negative and sine is positive. You do not need a calculator to know that. You need the picture.

If you ever walked around a circular track and tracked how far east or north you were from the center, you were already living the same idea. East–west is cosine. North–south is sine. When you are west of center, cosine is negative. When you are south of center, sine is negative.

One precise picture

Special angles have exact coordinates you can build from two familiar right triangles — the 30^{-60}-$90^{\wedge}$ and the 45^{-45}-$90^{\wedge}$ — and from the axes. On a unit circle the hypotenuse is 1, so the legs *are* the sine and cosine values.

On the unit circle:

- 0 or 0^: (1,0)
- π/6 or 30^: (32,1/2)
- π/4 or 45^: (22,22)
- π/3 or 60^: (1/2,32)
- π/2 or 90^: (0,1)

For an angle outside the first quadrant, find the reference angle — the acute angle to the nearest horizontal axis — get the absolute values from the reference, then attach signs by quadrant. Example: $2\pi/3$ is in quadrant II; its reference angle is $\pi/3$; so $2\pi/3$ = -1/2 and $2\pi/3$ = 32.

When you "unwrap" the circle — plot angle on the horizontal axis and the y-coordinate as height — you get the sine graph. Amplitude stretches or shrinks that height. Period stretches or shrinks how much angle it takes to repeat. For y = a(bx), the amplitude is |a| and the period is $2\pi/|b|$ when x is in radians. Degrees and radians are two names for the same turn; π radians is 180^. Confusing them is a mode error on the calculator, and a meaning error on paper.

The Pythagorean identity ^2θ + ^2θ = 1 is not a magic chant. On the unit circle it is x^2 + y^2 = 1 — the definition of the circle. Prefer structure rewrites after circle meaning. Do not open the week with an identity laundry list.

Domain and range talk belongs here when the course reaches inverse trig or when log and trig sit in the same function family week: sine's outputs stay between -1 and 1 unless amplitude changes that; a log's inputs stay positive. Name the restriction when it is in play. You do not need a full inverse-trig course to say "this output is impossible for plain sine."

Wrong answers you should be able to hear

1. **SOHCAHTOA as the entire course.** They can find an opposite side in a right triangle and freeze when the angle is $5\pi/6$ with no triangle drawn. Ask for the point on the circle. Draw it with them if needed. Say: "Opposite over hypotenuse still works — and on the unit circle the hypotenuse is 1, so the ratio *is* the coordinate."

2. **Calculator-first for special angles.** They punch ($2\pi/3$) and read a decimal before naming the point. Ask: "Where is that angle? What are the coordi-

nates?" Put the calculator away for special angles this week. Exact values matter for later algebra; decimals hide the structure.

3. **Wrong quadrant sign.** They get 32 for cosine of $2\pi/3$ — right absolute value, wrong sign. Ask which quadrant, then which coordinate is negative there. Have them point at the negative x-axis side of the circle.
4. **Degrees and radians mixed.** They leave the calculator in degree mode while the problem is in radians, or sketch a period of 2π as if it were 360 on a degree axis without converting. Ask them to say the measure both ways once before computing.
5. **Identity laundry list with no circle.** They recite ^2θ + ^2θ = 1 but cannot say why it is true on the unit circle. Ask them to point at x^2 + y^2 = 1. If they cannot, the identity is a slogan, not a reason.
6. **Sine sketch with no amplitude or period meaning.** They draw a wiggle and cannot say how high it goes or how long one cycle is. Ask: "What is the maximum y? How much input until it repeats?" Have them label both numbers on the sketch.
7. **Refuses reference-angle talk.** They try to memorize a separate coordinate for every angle instead of reducing to a first-quadrant value plus a sign. Show one angle reduced two ways — full memorization attempt vs reference-plus-sign — and compare the work.

Five-minute parent warm-up

Before the lesson, take a blank sheet. Draw a circle. Mark 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, and $2\pi/3$. Write coordinates for three of them from memory. Check against the list above. Say out loud: "Cosine is x. Sine is y." Convert one angle: $\pi/3 = 60^\circ$. That is enough. You do not need to be fluent in every identity to hear a wrong turn today.

If you feel rusty, that is normal. Many adults last saw this as a student. Five minutes with a blank circle beats twenty minutes of worrying that you are not "a math person."

How to teach it this week

This week's plan is the honest lesson plan — not a 180-day dump. One focus: unit-circle values and trig graphs (or right-triangle-to-circle transition at 15–16). Four or five sessions. One in-home try-it and one out-of-home try-it. Friday: one diagnostic wrong answer plus the done-enough checklist.

Session shape

Typical session: 45–75 minutes. Stretch to 90 if you are unwrapping the circle into a full sine graph for the first time.

1. **Warm-up (5 min).** One special-angle sine or cosine without a calculator. Example: $\pi/3$. Student places the angle, writes the coordinate, then the value. If yesterday's exit ticket missed a sign, warm up with that same angle again.
2. **Short model (8–12 min).** You build or complete a unit circle together. Place $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$. Say the definition once: cosine is the x-coordinate, sine is the y-coordinate. Then unwrap one point into a tiny sine sketch — angle on the horizontal, height y. Keep the model short. Do not narrate every identity you remember from high school.
3. **Student attempt (15–20 min).** Reference angle plus sign for one non-first-quadrant angle (for example $2\pi/3$ or $5\pi/6$). One application: a right-triangle height from an angle and a baseline, *or* a map bearing if that is this week's out-of-home try-it. They hold the pencil. You ask one question if they stall.
4. **One good question, then wait.** Use the locked opening below. Do not stack three questions at once.
5. **Mixed practice (10–15 min).** One special-angle value, one reference-angle-plus-sign, one parent-function transform of sine or cosine (amplitude or period), one degree–radian conversion. Mixed only when the new move has succeeded twice alone.
6. **Exit ticket (5 min).** Three coordinates on the circle and one one-period sine sketch with amplitude and period labelled. Date the page. Keep it.

Exact wording you can say:

- "Show me the point before the calculator value."
- "Reference angle first — then signs by quadrant."
- "Cosine is the horizontal coordinate. Sine is the vertical."

- "You may change your mind when we look at the circle."
- "As the angle grows through this quadrant, what is sine doing?"

How to fade help: Day 1 you draw the circle with them. Day 2 they draw; you only ask which quadrant. Day 3 they evaluate two special angles alone before you look. Day 4 they teach you one angle as if you forgot. When to stop talking: after you ask the locked question. A slow three-count is a convention from classroom think-time research, not a sacrament and not a math RCT. Look at the circle, not at their face, if the silence is hard. Do not grab the pencil. Do not finish the coordinate for them.

Sample week (precalculus type)

- **Monday:** Build special angles with coordinates from memory. Exit: four points labelled.
- **Tuesday:** Sine and cosine graphs from the circle unwrapped; amplitude and period. Exit: one sketch labelled.
- **Wednesday:** Map-trig or bearing out-of-home try-it; or law of sines if that is in-course this week. Notebook: diagram + equation + answer.
- **Thursday:** Mixed — values, signs, one transform, one conversion. Identities only as structure rewrites tied to the circle (for example Pythagorean from $x^{2+y}2=1$).
- **Friday:** Exit ticket; ramp angle or bridge height if not done Wednesday; run the done-enough checklist items you own this week.

Ages 15–16

Stay heavy on right-triangle trig with meaning, then introduce the unit circle as "the same opposite and adjacent, but now the hypotenuse is 1, so the ratios *are* the coordinates." Degrees first is fine; name radians as another measuring system and convert a few each week. Skip a full identity chapter this early. One Pythagorean identity from the circle is enough. Parent-function talk can stay light: "this sine goes twice as high" before formal a and b.

Out-of-home: ramp slope as rise/run *and* as an angle with , or a simple bridge/building height with a measured baseline. Safety: stay off roads; no climbing for a "trig photo"; no public quiz of a passerby.

If Foundations or Algebra I structure is still soft, label the repair and do it without

apology. Birthday did not finish Algebra I. The circle can wait a few weeks while linear and quadratic meaning get honest again — then return.

Ages 17–18

Expect full unit-circle fluency for special angles, reference angles in all quadrants, radians as the default for function graphs, amplitude and period with meaning, and parent-function transforms of sine and cosine. Identities as *structure rewrites* after circle meaning — not fifty formulas week one. Map trig / bearing and law of sines or cosines when the course is ready. Inverse trig and domain/range restrictions when the course includes them — always with a graph or circle in view.

Covariation language: "As θ increases through this quadrant, what is θ doing?" That sentence prepares ready students for Chapter 5 without dumping calculus. If they are already fluent on the circle and hungry for rate language, you may preview average rate of change of θ on a table — still precalculus grain, still not a birthday ticket into AP Calculus.

First try-it for the student (in-home)

Unit-circle dinner-table

- **Time:** 15–20 minutes.
- **Materials:** blank paper, pencil, a plate or compass for a circle (optional).
- **Safety:** ordinary kitchen or table — no heat, no sharp tools beyond a pencil.
- **The fun:** Place $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ on the circle and write coordinates. Then sketch a tiny sine wave from those y-values. Add one second-quadrant angle and argue about the sign over dinner.
- **The skill:** Coordinates as sine and cosine; signs by quadrant.

Parent functions still in the room

Precalculus is not only trig. The parent-function habit from Algebra II continues: linear, quadratic, polynomial, rational, exponential, logarithmic, trigonometric, piecewise. College Board's AP Calculus course map lists function properties, composition, algebra of functions, graphs, and sine/cosine from the unit circle among the prerequisites — labelled as a College Board map, not as a promise that

your kitchen will raise an AP score.[56] Use that list as a readiness mirror, not as a trophy.

When you transform y = x or y = x, name what changed: vertical stretch (amplitude), horizontal stretch (period), shift, reflection. Have the student sketch the parent lightly in pencil, then the transform in ink. The graph is the check. The equation is the claim.

Covariation toward calculus means sentences like these, spoken while looking at the circle or the sine graph:

- "As θ increases from 0 to $\pi/2$, the y-coordinate increases from 0 to 1."
- "As θ increases from $\pi/2$ to π, sine decreases from 1 to 0."
- "The height changes quickly near here and slowly near there — look at the steepness of the sine graph."

You are not assigning d/dθ(θ) yet. You are teaching eyes that notice change. That is the Thompson grain this book carries into Chapter 5: watch quantities change together before you crown a rule.[57]

Guardrails for this week

Open with the circle, not a fifty-identity packet. Treat any publisher scope or College Board map as a map, not a kitchen trial. If materials list AP Precalculus unit weights, treat them as that publisher's map — this book does not supply invented weights. A full BC-style series is optional, not required of every student. Keep bridge photos and map work on the ground and lawful. An eighteen-year-old who still needs right-triangle fluency before the circle gets that fluency without shame — place by skill.

Teach the circle. Require the written coordinate. Mix one out-of-home application

[56] College Board, *AP Calculus AB and BC Course and Exam Description* (effective Fall 2020): prerequisites include function properties, composition, algebra of functions, graphs, and sine/cosine from the unit circle — labelled College Board map in this book, not a kitchen RCT and not a promise that following its weeks raises AP scores. Access date for URLs in these notes: 5 September 2026. https://apcentral.collegeboard.org/courses/ap-calculus-ab/course.

[57] Patrick W. Thompson, "Images of Rate and Operational Understanding of the Fundamental Theorem of Calculus," *Educational Studies in Mathematics* 26, no. 2–3 (1994): 229–74; Thompson, Byerley, and Hatfield on coherence of meanings — covariation and watching quantities change together before rule flashcards. Teaching implication for this chapter's circle-to-sine and parent-function habit; calculus meaning is Chapter 5.

with in-home circle work. Keep the talk box honest: the opening question has no single recitation answer you are fishing for — you are fishing for a reason tied to a picture.

Practice that actually builds learning

Cars, maps, ramps, and bridges *are* the practice when they still force the circle, the diagram, and the written value. They do not replace the representation.

Talk box

Opening question (locked): "Where is this angle on the unit circle — and why is the sign what it is?"

Follow-ups:

1. **Meaning:** Reference angle first — then signs by quadrant. What is the reference angle for this one?
2. **Representation:** Show me the point before the calculator value. Mark it on the circle.
3. **Type / check:** Sketch one period of sine with amplitude and period labelled. What do those numbers mean on the graph?
4. **Revise:** You may change your mind. Try it another way — triangle inside the circle, or coordinates first.
5. **Written reason / units:** Convert this measure degrees $\leftrightarrow$ radians. Write one sentence: "Cosine is negative here because …"

How to wait: After you ask, wait a slow three. After they stop talking, wait again. If they are mid-reason, do not cut them off. Look at the circle or the sketch, not at the student, if the silence is hard.

What a stuck silence usually means: They are hunting SOHCAHTOA as the only tool; they want the calculator first; they are reciting an identity with no picture; wait-time 1 was zero; the angle is in a new quadrant and they have no reference-angle habit yet; they are guessing what you want instead of looking at the diagram. Point back at the circle. Do not fill the silence with the finished value. Do not turn the table into a fight.

Named try-its

1. Unit-circle dinner-table (in-home) — as above. Extend across the week: Monday first quadrant; Wednesday add $2\pi/3$ and $5\pi/6$ and force the sign talk; Friday unwrap to a full period sketch.

2. Reference angle + sign (in-home)

- **Time:** 10–15 minutes.
- **Materials:** scrap paper; calculator only for check *after*.
- **Safety:** none special.
- **The fun:** Evaluate $2\pi/3$ (or $7\pi/6$, or $5\pi/4$) without a calculator. Celebrate when the sign matches the quadrant — that is the win, not speed.
- **The skill:** Reference angle first, then quadrant sign.

3. Bridge / building angles (out-of-home)

- **Time:** 15–20 minutes.
- **Materials:** a safe baseline distance you can measure (sidewalk segment, parking-lot edge); a way to estimate or measure an angle of elevation (clinometer, phone angle app used carefully, or a cardboard sighting triangle).
- **Safety:** Stay on sidewalks and lawful spaces. Do not climb. Do not stop in traffic. Roads are for walking around, not for math theater. No lecture to a stranger's adolescent. No shoplifting a "test" of angles in a store aisle.
- **The fun:** Estimate a building, water tower, or bridge clearance height via of the elevation angle and the baseline. Name error sources out loud: baseline measurement, angle wiggle, "is the wall vertical?"
- **The skill:** Right-trig application with assumptions named. Write the equation before the number. Bring the notebook — the diagram is the work.

4. Map trig / bearing (out-of-home)

- **Time:** 15–20 minutes.
- **Materials:** a trail map, park map, or printed street map with a scale; pencil and ruler.
- **Safety:** Stay put while doing the math — picnic table, car, or visitor center. Do not navigate by half-finished trig into unsafe terrain. Ordinary outdoor courtesy.
- **The fun:** Pick two points. Find distance and a bearing or direction using the map scale and a protractor or careful estimate. If ready, break into east–north components, or use law of sines/cosines on a non-right triangle formed

by three landmarks.
- **The skill:** Non-right application at precalculus grain; scale and angle together; Precalc-level written work.

5. Ramp slope as rise/run and as angle (out-of-home)

- **Time:** about 10 minutes.
- **Materials:** a wheelchair ramp, driveway, or playground incline you can measure safely with a tape and a level or a book as a "horizontal."
- **Safety:** Do not block accessibility. Measure from the side. No climbing under railings. Leave the ramp clear for people who need it.
- **The fun:** Compute rise/run (Algebra I slope) and the angle whose tangent is that ratio. Same object, two languages. Optional: compare two ramps.
- **The skill:** Covariation bridge from slope to θ; continuity with Algebra I.

Composite illustration (labelled)

A trail-map bearing with made-up distances is fine if you label it as an illustration, not as a reported family trip. A bridge-height estimate with a phone angle app is fine if you name measurement error. Pharmacy-style "angle of a ramp for accessibility" talk stays geometry — not medical or legal advice about codes. Keep stakes ordinary. No live controversy as a trig prompt.

Blocked practice (new move)

While the move is new, give only reference-angle-plus-sign items, or only special-angle placements — not a mixed identity blizzard. Blocked practice builds the habit. Five similar items with the circle in view beat one of each type while the move is fragile.

Mixed practice (when to use it)

When they can place three special angles and evaluate one second-quadrant cosine without you, mix: one special angle, one reference-plus-sign, one amplitude/period sketch, one degree–radian conversion, one right-triangle application. Revisit Tuesday's angle on Thursday. Life-of-the-habit is session rhythm — revisit, mix, stop — not a ninth chapter.

One incorrect example to diagnose

Student writes $2\pi/3 = 1/2$ and looks satisfied.

What it usually means: they found the reference value and skipped the quadrant. Ask: "Which quadrant is $2\pi/3$? Is the x-coordinate positive or negative there?" Do not say only "wrong" and walk away. Name the missing move: reference first, *then* sign. Have them correct the same item and explain the fix in one sentence.

A second common miss: student sketches y = x with peaks at height 2 and says "that's just how sine looks." Ask what amplitude means. Have them compare y = x and y = 2 x on the same axes.

For the student

Trigonometry is not only "opposite over hypotenuse."

On a circle of radius 1 centered at the origin, every angle lands on a point (x,y). Cosine is x. Sine is y. That is the definition that lets sine and cosine be functions of any angle — not only acute angles in right triangles.

SOHCAHTOA still works inside a right triangle. On the unit circle the hypotenuse is always 1, so opposite/hypotenuse *is* the y-coordinate, and adjacent/hypotenuse *is* the x-coordinate. Same idea. Bigger playground.

Tiny worked example

Find $\pi/3$ and $2\pi/3$ without a calculator.

- $\pi/3$ is 60^ in the first quadrant. The point is (1/2,32). So $\pi/3 = 32$.
- $2\pi/3$ is in the second quadrant. Reference angle: $\pi/3$. Absolute values: 1/2 and 32. In quadrant II, x is negative and y is positive. So $2\pi/3 = -1/2$ and $2\pi/3 = 32$.

Notice: you did not need a calculator. You needed the picture and the quadrant.

Two tries

1. Place $\pi/4$ and $5\pi/6$ on a blank unit circle. Write coordinates for both. Check the signs for $5\pi/6$ — which coordinate should be negative?

2. Sketch one period of y = 2 x (radians). Mark the maximum height and where the graph completes one cycle. What is the amplitude? What is the period?

Explain it back

In one or two sentences, tell someone: what cosine means on the unit circle, and why cosine is negative in the second quadrant. Use the words “coordinate” and “quadrant.”

Challenge

Unwrap the unit circle into a sine graph for 0 to 2π: plot at least the axis and special-angle points you know, then connect them smoothly. Label amplitude and period. Then write one sentence of covariation: “As θ goes from 0 to $\pi/2$, θ …” and another: “As θ goes from $\pi/2$ to π, θ …”

Optional stretch: convert 150^ to radians, place it, and evaluate sine and cosine exactly.

You are allowed to wait. You are allowed to change your mind when you look at the circle. You are allowed to be good at this without being the fastest person in the room.

More tries if you finish early

3. Without a calculator, evaluate $7\pi/6$ and $7\pi/6$. Write the reference angle, the quadrant, and both values.
4. On one set of axes, sketch y = x and y = (2x) for one or two periods. What changed — amplitude or period? Explain in a sentence.
5. A ramp rises 1 foot for every 12 feet of run. What is anθ? Estimate θ with a calculator *after* you write the ratio. Connect this to slope from Algebra I.

If it isn’t clicking

No shame. Hearing a wrong answer, asking a better question, and naming a better move is teaching. Silence in the face of a keyword error is not kindness.

Diagnostic 1: They only have SOHCAHTOA.
Next move: Draw a right triangle *inside* the unit circle for $\pi/6$. Hypotenuse is 1. Opposite is sine. Adjacent is cosine. Say: "Same ratios — and now the hypotenuse is always 1, so the ratios *are* the coordinates." Stay there for a few days before new quadrants. Do not assign a packet of identities yet.

Diagnostic 2: Calculator-first; special angles wrong or opaque.
Next move: Calculator away for a week on special angles. Build a personal unit-circle reference card by hand (not a printout they never made). Exit ticket each day: three values with no device. When they earn the calculator back, it is for non-special angles and for checking graphs — labelled.

Diagnostic 3: Signs and radians scramble together.
Next move: Separate the skills for two sessions. Session A: signs only, all in degrees if needed, four quadrants. Session B: convert five angles both ways, then redo one cosine in radians. Do not pile identities on top until both sessions stick.

When to slow down: They cannot place $\pi/3$ without looking after several days of circle practice. Return to first-quadrant building from the 30-60-90 triangle. Fewer angles, more talks.

When to go ahead: They evaluate two non-first-quadrant values correctly, sketch one period with amplitude and period, and convert degrees $\leftrightarrow$ radians for angles in play. Add transforms and applications. Ready students may taste covariation tables toward Chapter 5.

When to get a human tutor: Anxiety about "trig" has frozen the attempt for two weeks despite circle practice, or every session becomes a fight about the calculator. A short burst of coached sessions on the circle often unsticks more than another identity worksheet. A tutor is a tool, not a verdict on you or them.

Correct a wrong answer without crushing the attempt: "The absolute value matches the reference. The sign is the next move — which quadrant?" Never correct the student is not this book's rule. Crushing the attempt is not the goal either.

A note on materials fit (not a sales pitch)

Kitchen and notebook first: blank unit circles, dated exit tickets, OpenStax-style open texts or Khan practice *after* the attempt if you use them. If you use a video course (Teaching Textbooks, CTCMath, Derek Owens, Thinkwell) or a rigorous

text path (AoPS Precalculus, Foerster-style, Saxon Advanced Math as *materials*), the credit title a stranger can read is still **Precalculus** when the year's work matches — never the brand alone. Placement diagnostics from publishers are tools for noticing gaps, not laws. Use them to notice. No single program needs crowning from a school rating.

If Algebra II function families are unfinished, finish them with meaning before you drown in trig identities. Critical Foundations continuity still applies when fractions, ratio, or linear structure are soft — birthday did not fix them.

Tools, including AI

Optional helpers for you, the adult. Keep this short. The full AI box lives in the math hour chapter — point back there for COPPA-off rules, co-hold defaults, unaided-first, SCRIPT-after-try, and the photo-to-key ban.

After the student has tried:

- **Desmos or GeoGebra** to check a sine graph's amplitude and period, or to explore what a and b do in y = a(bx) — labelled, eyes open. The student should already have a hand sketch.
- **Scientific or graphing calculator** in the correct mode (radians vs degrees) after special-angle work is done by hand. Mode discipline is part of the lesson.
- **Wolfram|Alpha or a CAS** to verify an identity rewrite *after* they have a reason from the circle — not to generate the rewrite as their work.

You may, on your account after they try: ask a tool to explain the unit circle *to you* from today's angles; generate isomorphic practice with the answer key held by you; get a short SCRIPT for what you will say next; diagnose a cropped photo of the paper (not their face).

You may not: paste the worksheet for completion; let them ask "what's the answer" of a window during the attempt; photo-to-key (Photomath, Mathway, Symbolab, or any scan-and-solve) so a solution pops up for the exact problem; park an unsupervised chatbot as the only partner during the attempt — not at 15, not at 18 — when you are co-teacher; write their free-response for them.

For ages 15–16, stay in the room and co-hold. For ages 17–18, more agency still does not mean a secret math friend during the try.

One on-age caution, then teach: in a high-school math field experiment (grades 9–11, nearly 1,000 students, Turkey), an unguarded chatbot made practice look better and left students worse on the test with the window closed; a guarded tutor that withheld the answer left the unaided test about the same as no AI (Bastani et al., 2025). That is this age band's math trial — not a U.S. homeschool RCT. The model may prepare you and the next problem; it may not do their problem during the attempt.[58]

What "done enough" looks like

Before moving on from a precalculus / trigonometry block, check skill — not birthday:

- ☐ Evaluates special angles (multiples of $\pi/6$ and $\pi/4$ in common use) for sine and cosine without a calculator.
- ☐ Finds a reference angle and attaches the correct quadrant sign.
- ☐ Sketches one period of sine or cosine with amplitude and period labelled and explained in plain words.
- ☐ Converts between degrees and radians for angles in play.
- ☐ Solves a right-triangle application *and* (when the course includes it) one non-right application (map, bearing, or law of sines/cosines) with the diagram and equation shown.
- ☐ Can say in plain words: cosine is the x-coordinate on the unit circle; sine is the y-coordinate.
- ☐ Connects at least one identity to the circle (for example Pythagorean as $x^{2+y}2=1$), not only as a memorized string.
- ☐ Uses covariation language once: as the angle changes, what is the coordinate doing?
- ☐ Notebook shows dated exit tickets and at least one in-home plus one out-of-home try-it for the unit.

[58] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman, "Generative AI without guardrails can harm learning: Evidence from high school mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. Turkish grades 9–11 field experiment; unguarded chatbot +48% practice / −17% unaided exam; guarded tutor +127% practice / unaided ≈ control. ON-AGE for this book — not a U.S. homeschool RCT. Kitchen-table rule: the model may prepare the adult and the next problem; it may not do the student's problem during the attempt.

A “grade 11 precalculus workbook” is a publisher’s scope, not a legal grade, and not a transcript line. If the year’s work was Precalculus, the stranger-readable title is **Precalculus**. Never a brand. Never Emerging Adults Math I.[59]

Done-enough is not “fifty identities memorized.” Done-enough is circle meaning, function graphs with amplitude and period, and applications that still show the representation. Identities as structure rewrites can continue as the course goes — after the circle is real.

If calculus is next for a *ready* student, this chapter’s covariation habit is the bridge. Readiness is function fluency (including trig), comfort with algebraic structure, and capacity to reason about changing quantities — not an eighteenth candle. If Statistics or a stronger Algebra II finish is next, that can still be excellent. Many strong high-school completions end in rigorous Precalculus plus Statistics. Place by skill.

Revisit Tuesday’s angles on Thursday. Mix last week’s signs with this week’s graphs. Stop when the exit ticket is honest. That rhythm *is* the life of the habit — it lives in the math hour, in this checklist, and in the records you keep — not in a ninth teaching chapter.

If the out-of-home try-it goes poorly because of weather or access, swap in a second in-home circle session the same week and reschedule the ramp or map for the next clear day. Consistency of the representation matters more than forcing a field trip on a bad afternoon.

Keep the pencil in their hand. Your job is the question, the wait, and the circle. Their job is the coordinate, the sign, and the sketch. That division of labor is the math hour applied to trigonometry.

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[59] Ellena Sempeles and Jiashan Cui, *Parent and Family Involvement in Education: 2023*, NCES 2024-113 (Washington, DC: National Center for Education Statistics, September 2024): no national homeschool diploma. Transcript titles in this book are Algebra I, Geometry, Algebra II, Precalculus, Calculus, or Statistics by content — never a brand, never “Emerging Adults Math I.” National Mathematics Advisory Panel, *Foundations for Success* (2008), Finding 15: place by skill; prior opportunities to learn matter more than age-stage slogans.

Chapter 5 — Calculus with meaning

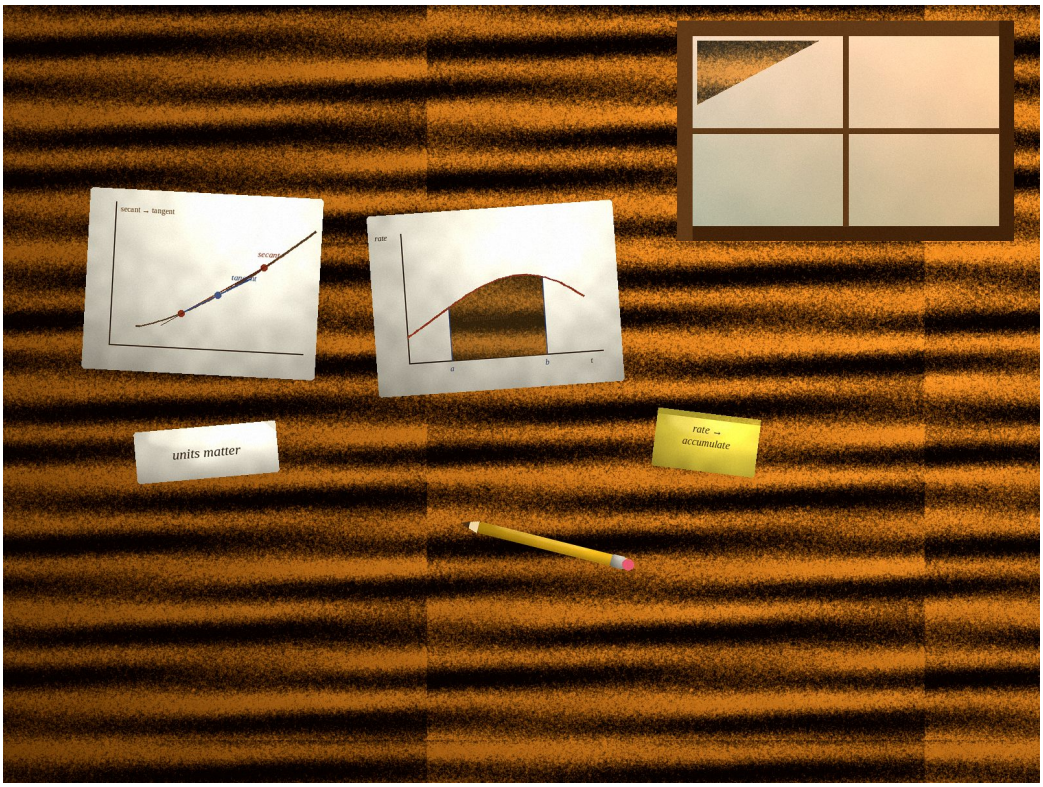

Figure 8: A kitchen-table still-life: a rate graph with a shaded accumulation region; a table of shrinking average rates; units written in the margin. No people. No logos.

Why this matters

Calculus is optional for many strong high-school paths. It is powerful for students who are *ready*.

Ready means fluency with functions (including trig and exponential or log as the course needs), comfort with algebraic structure, and capacity to reason about changing quantities. Ready does not mean an eighteenth birthday. Place by skill, not candle. A fifteen-year-old who clears precalculus readiness may begin calculus meaning with you beside them. A seventeen-year-old who still treats every derivative as a memorized rule with no rate story is not behind a birthday — they need meaning first, or more precalculus, or a strong Statistics year. Many excellent completions end in rigorous Precalculus plus Statistics, or Algebra II plus Statistics, without calculus. That can still be excellent. Do not make calculus the virtue signal of this manual.

When calculus *is* the work, refuse "power rule week" with no rate story. The conceptual spine this chapter follows is limits and continuity $\rightarrow$ derivative as instantaneous rate of change $\rightarrow$ applications with meaning $\rightarrow$ integral as accumulation $\rightarrow$ Fundamental Theorem as relationship. That order matches how changing quantities actually behave. Procedures come after meaning — they do not replace it.

College Board's AP Calculus Course and Exam Description (Effective Fall 2020) is a useful *map* when a family chooses an AP-aligned path: Big Ideas Change (CHA), Limits (LIM), and Analysis of Functions (FUN); Unit 1 Limits and Continuity; Unit 6 Integration and Accumulation of Change; multirepresentational work — graphically, numerically, analytically, verbally. Label it a College Board map. It is not a kitchen RCT, and it does not prove that following its weeks at home raises AP scores.[60]

Parent learns this week: hear power-rule-first, and ask what the average rate is becoming and in what units. Student tries today: compute average rates on shrinking intervals, estimate a derivative with units, or interpret area under a rate graph as

[60] College Board, *AP Calculus AB and BC Course and Exam Description* (effective Fall 2020): Big Ideas Change (CHA), Limits (LIM), and Analysis of Functions (FUN); Unit 1 Limits and Continuity; Unit 6 Integration and Accumulation of Change; multirepresentational work graphically, numerically, analytically, verbally. Labelled College Board map — not a kitchen RCT and not proof that following its weeks at home raises AP scores. Access date for URLs in these notes: 5 September 2026. https://apcentral.collegeboard.org/courses/ap-calculus-ab/course.

accumulation in one sentence.

For the parent: understand it yourself

You do not need to re-earn a college calculus credit to teach the opening weeks well. You do need to hear power-rule flashcards with no rate story, a confusion between where f' is zero and where f is zero, an area under a speed graph named in the wrong units, and a birthday offered as readiness — and to ask a better question without taking the pencil.

One everyday picture

A car's trip computer shows average speed for the last few minutes. As you shrink the time window — last minute, last ten seconds, last second — that average is hunting a number that describes how fast you are going *right now*. Instantaneous rate is what average rate becomes as the interval shrinks. Units matter: if distance is in miles and time in hours, the rate is miles per hour. If the rate itself is changing, you may be talking about acceleration in miles per hour per hour.

Now flip the story. The speedometer needle over time draws a graph of speed versus time. The area under that graph accumulates distance. Finger-trace the region from time a to time b: that accumulation is how far you went. Integral language begins as "running total of a rate," not as "antiderivative recipes only."

One precise picture

Limit. As x approaches a, what value does f(x) approach? From a table: plug in values closer and closer to a from both sides when you can. From a graph: read the y-height the curve is heading toward. Continuity roughly means the limit matches the function value — you can draw through the point without lifting the pencil. Limits come before derivative rules so the student has language for "getting arbitrarily close."

Derivative as rate. Average rate of change of y with respect to x on [a, a+h] is f(a+h)-f(a)/h. As h shrinks, that quotient becomes the derivative f'(a) when the limit exists — instantaneous rate at x = a. On a graph, secant lines through (a, f(a)) and a nearby point get steeper or flatter toward the tangent. Always ask: in what units?

Integral as accumulation. If f is a rate (gallons per minute, miles per hour, dollars per day), then _a^b f(x),dx accumulates the total change in the underlying quantity from a to b. Riemann sums are the honest numerical story: add many thin rectangles (or other slices) of rate $\times$ tiny time. The definite integral is what those sums approach.

Fundamental Theorem (in words first). Accumulation and rate talk to each other. If F(x) = _a^x f(t),dt, then F'(x) = f(x) under the usual conditions — the rate of change of the accumulation function is the integrand. And _a^b f(x),dx = F(b)-F(a) when F' = f. Say it in words before you rush to antiderivative recipes. Procedures after meaning; meaning is not a license to skip procedures forever.

Thompson's line (rate, accumulation, FTC as relationship) is the grain of this chapter.[61] Tallman and Carlson's sampling of Calculus I finals found few items that asked students to go beyond procedures — a reminder that a kitchen which only drills rules may match a weak exam culture, not a strong understanding.[62] Use that as sobriety, not as a scare.

Limits before rules — a longer look

A limit is not a mystical fog. It is a disciplined answer to "what are we approaching?" Give three representations every week you teach limits:

- **Table:** x values 2.1, 2.01, 2.001 and 1.9, 1.99, 1.999 when studying x 2. Write f(x) beside them. Estimate.
- **Graph:** Trace with a finger toward x = 2. Read the height the curve is heading for — which may differ from a hole's missing y.
- **Algebra:** Simplify a rational expression and then substitute when the simplified form is continuous there. Algebra is a tool *after* the numerical and graphical hunt have made the question real.

[61] Patrick W. Thompson, "Images of Rate and Operational Understanding of the Fundamental Theorem of Calculus," *Educational Studies in Mathematics* 26, no. 2–3 (1994): 229–74. Rate, accumulation, and FTC as relationship — the conceptual spine this chapter follows before power-rule flashcards.

[62] Tallman and Carlson's sampling of Calculus I final exams (cited in Patrick W. Thompson, Cameron Byerley, and Neil Hatfield, "A Conceptual Approach to Calculus Made Possible by Technology," *Computers in the Schools* 30, no. 1–2 (2013): 124–47): few items asked students to go beyond procedure recall — sobriety for a kitchen that only drills rules, not a scare about every exam.

Continuity talk in one kitchen sentence: "The limit matches the function's value, so the graph doesn't break there." Discontinuity: hole, jump, vertical asymptote — show each once on a sketch. You do not need a topology course. You need the student to stop saying "plug in" as the only limit strategy when plugging in fails.

Rate language that travels

Write units on every rate. Distance/time → speed. Velocity/time → acceleration. Volume/time → fill rate. Cost/item → marginal thinking in a light, ordinary-stakes way (no stock-tip sermon). Related rates later in a course still begin with a sketch and labelled rates with units — the same habit as this chapter's opening week.

Covariation from Chapter 4 pays rent here: as input changes, output changes, and the *rate* of that change may itself change. A linear function has constant rate. A quadratic's average rates change in a patterned way. Trig and exponential rates wait until those functions are fluent.

Accumulation that is not only "area of a shape"

Students who only hear "integral = area" miss the story when the integrand is a rate. Prefer: "This graph shows gallons per minute; the region accumulates gallons." Area is the geometric shadow of accumulation when the axes mean what they mean. Net change can be positive minus negative regions when rate dips below zero — introduce that when the course is ready, still with units.

Riemann sums by hand with four or eight rectangles beat a calculator integral button on day one of accumulation. The button can check later. The hand sum teaches the definition's spirit.

Applications with meaning (not a theme-park of tricks)

When the course reaches max/min or motion, keep the rate sentence first. A critical point where f' = 0 or is undefined is a *candidate* — test it. Motion: s, v = s', a = v' with units. Do not let related-rates or optimization become keyword hunts ("the word *maximum* means set derivative to zero" without a sketch). Schema: draw, label rates/units, write related equation, differentiate, interpret.

Wrong answers you should be able to hear

1. **Power-rule flashcards with no rate story.** They chant "derivative of x^n is nx^n-1" and cannot say what f'(2) means in a distance–time story. Ask: "As the interval shrinks, what is this average rate becoming — and in what units?"
2. **Confuses zeros of f' with zeros of f.** They say the function is zero where the derivative is zero. Ask them to sketch a hilltop: height is not zero; slope is.
3. **Area under a rate graph in wrong units.** Speed graph in miles per hour; time in hours; they call the area "miles per hour" instead of miles. Ask what rate $\times$ time produces.
4. **FTC as antiderivative recipes only.** They can compute x^2,dx and cannot say that an accumulation function's derivative recovers the integrand. Ask for the relationship in words.
5. **Calculator replaces no-calculator fluency.** They graph everything and cannot estimate a limit from a table or a derivative from secants by hand. Require one unaided numerical or graphical item each session.
6. **Birthday offered as readiness.** "I'm eighteen, so I should be in calculus." Reply with skill: functions, structure, changing quantities. Offer Precalculus repair or Statistics with pride if the gate is not cleared.
7. **Insists everyone must finish BC.** Name Precalculus + Stats as a strong completion. AP Calculus BC is an option for some ready students, not a moral requirement.

Everyday pictures that stay ordinary

Kitchen timer and oven temperature rising: average degrees per minute on shrinking intervals. Walking pace on a path you already use: time a segment, then a shorter segment. Water bottle fill from a tap: rough gallons per minute (composite estimate — not a plumbing code). Keep stakes ordinary. No live controversy. No stock tips as sermons. No medical advice.

These pictures motivate. They do not replace the table of difference quotients or the shaded accumulation region in the notebook. The representation still has to

appear.

Precise derivative definition (parent cheat sheet)

$$f'(a) = \lim_{h \to 0} f(a+h) - f(a)/h$$

when the limit exists. Say it: "the limit of average rates as the change in input goes to zero." Alternate form with x a is fine when your text uses it. The power rule for f(x) = x^n falls out as efficiency later: f'(x) = nx^n-1 for nice n you have proved or accepted in-course. Efficiency without the limit story is what this chapter refuses as the *only* meaning.

Product, quotient, and chain rules wait until rate meaning and power-rule efficiency are real — still taught as rate stories when you can ("how fast is a product changing?"), still practiced as procedures after.

Precise integral / FTC (parent cheat sheet)

Definite integral as limit of Riemann sums; notation _a^b f(x),dx. Accumulation function F(x) = _a^x f(t),dt. FTC Part 1 idea: F'(x) = f(x). FTC Part 2 idea: _a^b f = F(b)-F(a) when F' = f. Your text's numbering may swap "parts"; the relationship matters more than the label. Antiderivative practice is the computational twin — after the words.

Five-minute parent warm-up

Make a tiny table: t = 0,1,2,3 seconds; position s = 0, 4, 10, 18 meters. Compute average rates on [0,3], [1,3], and [2,3]. Say out loud what is happening to the average as the interval shrinks toward t = 3. Name the units: meters per second. That is today's idea. You do not need the power rule to hear a wrong turn.

How to teach it this week

Only for ready students. If readiness is soft, stay in Chapter 4 or run a Statistics block (Chapter 6). Re-check midyear; pride is not a placement instrument.

Friday habit: one diagnostic wrong (power-rule-without-units, or area in wrong units) plus the done-enough checklist items you own this week. Correct the wrong without crushing the attempt. Date the page.

This week's plan is the honest lesson plan. One focus for a rate week *or* an accumulation week — do not smash both into Monday. Four or five sessions. One in-home and one out-of-home try-it. Friday: conceptual plus procedural exit ticket.

Session shape

Typical session: 45–75 minutes. Use 90 when a limit investigation or accumulation sketch needs room.

1. **Warm-up (5–8 min).** Average rate from a table with units written. Or: estimate a limit from four nearby x-values.
2. **Short model (10–15 min).** Rate week: show secants on a graph shrinking toward a tangent; *then* name the derivative; *then* a rule if the day calls for it. Accumulation week: speed-vs-time sketch; finger-trace area; name what accumulates; *then* a Riemann sum sketch; *then* FTC words.
3. **Student attempt (15–25 min).** Shrinking-interval average rates with units, *or* interpret shaded area under a rate graph in one sentence plus a rough numeric estimate.
4. **One good question, then wait.** Locked opening below.
5. **Mixed practice (10–15 min).** One limit-from-table, one rate interpretation, one routine derivative or sum *after* meaning that day, one "what are the units?" prompt.
6. **Exit ticket (5–8 min).** Conceptual prompt + procedural prompt. Date it. Keep it.

Exact wording you can say:

- "As the interval shrinks, what is this average rate becoming — and in what units?"
- "Show me the secant on the graph. Now a closer one."
- "If this graph is speed, what does the area accumulate?"
- "State the Fundamental Theorem in words linking accumulation and rate."
- "You may change your mind when we look at the table."

How to fade help: Day 1 you draw secants with them. Day 2 they draw; you only ask for units. Day 3 they estimate f'(a) from a table alone before any rule. Day 4

they explain FTC in words to you as if you forgot. When to stop talking: after the locked question. Wait a slow three; wait again after they pause. Look at the graph or table, not at their face, if silence is hard. Do not grab the pencil. Do not paste a machine's solution as theirs.

Rate week (sample)

- **Monday:** Average rate tables; units every time. Exit: three averages + units.
- **Tuesday:** Secants $\rightarrow$ tangent talk; derivative as instantaneous rate; definition language. Exit: estimate f'(a) from a graph.
- **Wednesday:** Sports or physics rates out-of-home (splits as averages; optional crude derivative preview). Notebook: rate vs total.
- **Thursday:** Power rule and other routine rules *after* meaning — efficiency now that rate is real. Mix with a limit-from-table.
- **Friday:** Conceptual + procedural exit; done-enough items you own this week.

Accumulation week (sample)

- **Monday:** Speedometer composite illustration; area as distance. Exit: one sentence naming units of the area.
- **Tuesday:** Riemann sum sketch by hand; left/right rectangles qualitatively.
- **Wednesday:** Physics-lab slope or sports accumulation talk if available; else in-home tank-filling or cost-accumulation story (composite, ordinary stakes).
- **Thursday:** FTC in words; then antiderivative practice tied back to accumulation.
- **Friday:** Exit ticket linking rate and accumulation; checklist.

Ages 15–16

Calculus meaning only if precalculus readiness is clear — function fluency, trig as needed, structure, hunger for changing quantities — with close supervision. Many 15–16 students belong in Algebra II or Precalculus this year. If you begin limits and average-rate language early, keep the dose small: tables and graphs before symbol recipes. Do not race toward an AP exam because a sibling did.

Out-of-home: sports splits as average rates; a safe physics demo of position vs

time if you have lab access. Safety: ordinary sports and lab rules; no dangerous "acceleration experiments" on public roads.

Ages 17–18

This is the common band for a first calculus course when readiness holds. Expect multirepresentational work: graph, table, equation, verbal. Use the College Board map if you are AP-aligned — still labelled a map. Stronger students may go deeper into applications and, if appropriate, BC topics; readiness and motivation warrant acceleration, not birthday. Students who clear precalculus late may start calculus meaning midyear without shame. Students who do not clear readiness do Precalculus or Statistics honestly.

Multirepresentational habit (CED map grain)

College Board's practices ask students to move among graph, table, analytic work, and verbal explanation. Steal the habit without worshipping the brand:

1. Same prompt, four boxes on one notebook page: Graph | Table | Equation | Words.
2. Student fills at least three. The fourth can be partial.
3. Parent asks: "Do they agree?" Disagreement is the lesson.

Example prompt: "Estimate the derivative of f at x = 3." Graph: tangent sketch. Table: difference quotients. Equation: rule if known. Words: units and meaning. This is AP-aligned practice when you want it — and good mathematics when you do not sit the exam.

How hard to push procedures

After two solid rate weeks, procedural fluency belongs. Brief mixed sets of derivatives you have taught — accurate first, not a timed calculus kit. Same for antiderivatives after accumulation weeks. Fluency without meaning was the problem; meaning without any fluency becomes a different problem by December. Conceptual understanding, computational fluency, and problem solving belong together.

Out-of-home safety reminder

Sports meets and labs are privileges. Stay in spectator or student roles. Do not bring drones into restricted spaces for a "rate video." Do not experiment with cars on public roads for acceleration data. Parking-lot physics only where lawful and safe. The notebook sketch from a published split time is enough.

First try-it for the student (in-home)

Average rate → shrink interval

- **Time:** 15–20 minutes.
- **Materials:** scrap paper; a distance/time or height/time table (hand-made is fine).
- **Safety:** none special.
- **The fun:** Hunt the number the averages are approaching. Argue about units.
- **The skill:** Derivative as instantaneous rate (Thompson grain) before any power-rule chant.

Materials and maps (not a sales pitch)

If you use OpenStax Calculus, Khan practice after attempts, Derek Owens, Thinkwell, AoPS Calculus, or Saxon Calculus as *materials*, the transcript title is still **Calculus** when the year's work matches — never the brand. College Board CED is a scope map for AP-aligned families. CLEP Calculus exists as an option with caveats (see records); CLEP is not NCAA core. Dual enrollment calculus is a state-and-college gate, not a right this book can grant.

Kitchen $0 start: notebook, hand tables, hand sketches. Technology after meaning.

Guardrails for this chapter

There is no conceptual-calculus home trial this book can quote as a kitchen guarantee. The College Board map does not prove that following its weeks raises AP scores at your table. Calculus is for ready students — not every reader. Thompson's grain is rate and accumulation first, then procedures — not "skip procedures forever." There is no kitchen AI coefficient to paste onto homework. Keep physics-rate videos on lawful ground. Sports stats stay rates — not identity fights.

Pharmacy dosage as rate, if you ever use it as a composite illustration, is **not medical advice**.

Practice that actually builds learning

Sports rates and physics slopes motivate. They do not replace the graph, the table, or the units sentence.

Talk box

Opening question (locked): "As the interval shrinks, what is this average rate becoming — and in what units?"

Follow-ups:

1. **Meaning:** What are we hunting — a slope, a speed, a rate of fill? Say it in plain words.
2. **Representation:** Show the secant on the graph. Draw a closer one. Or show the rows on the table.
3. **Type / check:** What are the units of dy/dx here? Write them beside the number.
4. **Accumulation link:** If this graph is speed, what does the area accumulate? Name units of the area.
5. **Revise / FTC:** State FTC in words linking accumulation and rate. You may change your mind; try another representation.

How to wait: Ask, then a slow three. After they stop, wait again. If mid-reason, do not cut off. Look at the graph or table.

What a stuck silence usually means: Power-rule flashcards with no story; wait-time 1 was zero; they confuse zeros of f' with zeros of f; they want the calculator to finish the thought; units were never named; the table values are too unfamiliar. Point back at the representation. Do not fill silence with the finished derivative.

Named try-its

1. Limit from a table (in-home)

- **Time:** 15 minutes.

- **Materials:** scrap table as x a from both sides when possible.
- **Safety:** none special.
- **The fun:** Estimate the limit; compare to plugging in x = a (continuity talk — when they match, when they do not).
- **The skill:** Limit language before rules.

2. Average rate → shrink interval (in-home) — as above. Extend: three nested intervals; estimate f'(a); write units.

3. Speedometer accumulation (composite illustration, in-home)

- **Time:** 15–20 minutes.
- **Materials:** hand-sketch of speed vs time (made-up but realistic numbers are fine; label as illustration).
- **Safety:** none special — this is paper, not driving advice.
- **The fun:** Finger-trace the area. Ask: miles accumulated, or gallons, or what? Match units to the story.
- **The skill:** Integral as accumulation; then Riemann language; then FTC words.

4. Sports / physics rates (out-of-home)

- **Time:** during a meet, practice, or lab period you already attend.
- **Materials:** split times, box score, or lab position data; notebook.
- **Safety:** ordinary event rules. Do not interfere with athletes or equipment. No identity fights — sports stats are rates.
- **The fun:** Splits as average rates; compare two intervals; optional crude sense of "right now" vs "over the whole race."
- **The skill:** Rate vs total; units; calculus preview without requiring AP.

5. Physics-lab slope (out-of-home)

- **Time:** if you have lab access (co-op, dual enrollment, home lab kit).
- **Materials:** position–time data; graph paper or plotted printout.
- **Safety:** follow lab safety. No unauthorized equipment use.
- **The fun:** Slope of position–time as velocity; talk residuals or measurement noise briefly.
- **The skill:** Calculus preview with real messy data; units.

Blocked practice

New move week: only shrinking averages, or only limit tables, or only area-under-rate interpretations — not a mixed AP free-response dump on day two.

Mixed practice

When they can estimate one derivative from a table with units and interpret one accumulation area, mix: limit, rate, routine derivative taught, accumulation sentence, units check.

One incorrect example to diagnose

Student computes d/dx(x^3) = 3x^2, then says of a position function s(t) = t^3 that "the derivative being zero means the car is at position zero."

What it usually means: rule without rate story; zeros of f' confused with zeros of f. Ask for units of s'(t). Ask for a sketch. Ask what s'(t) = 0 means for motion (momentarily stopped — position may be anywhere).

Another miss: shaded region under v(t) in m/s vs seconds labelled "m/s." Ask what rate $\times$ time produces.

For the student

Calculus begins with a honest question about change.

Average rate asks: how much did this quantity change per unit input on this interval? Instantaneous rate asks: what is that average becoming as the interval shrinks? Accumulation asks: if I keep adding up a rate over time, what total do I build?

Tiny worked example

Positions at times t = 1, 2, 3 seconds: s = 2, 6, 14 meters.

- Average rate on [1,3]: 14-2/3-1 = 6 m/s.
- Average rate on [2,3]: 14-6/3-2 = 8 m/s.

As we shrink toward t = 3, the average is rising in this table. If we had closer data, we would hunt a number near the instantaneous rate at t = 3, still in meters per second. That hunting is derivative thinking — before any power rule.

Two tries

1. Make or use a table for y = x^2 near x = 2. Compute average rates on three shrinking intervals. Estimate y’ at x = 2. Units if x is seconds and y is meters?
2. Sketch a speed graph that stays at 40 miles per hour for 2 hours. What does the area represent, and what is its value?

Explain it back

In two sentences: what a derivative means as a rate (include units talk), and what area under a rate graph accumulates.

Challenge

State the Fundamental Theorem in your own words so that a parent who last saw calculus years ago can hear the relationship between accumulation and rate. Then compute one routine antiderivative and explain how it connects to your sentence.

Optional: from a hand-drawn curve, estimate f’(a) with a tangent sketch and compare to a difference quotient with small h.

You are allowed to wait. You are allowed to change your mind when the table gets denser. You are allowed to take Statistics or Precalculus instead of rushing calculus — readiness beats birthday.

More tries

3. From a hand-drawn curve through (1,2), draw a secant to (1.5, 2.8) and a closer secant of your choosing. Which looks more like the tangent at x = 1? Estimate the slope.
4. A tank fills at 3 gallons per minute for 10 minutes, then at 1 gallon per minute for 5 minutes (composite illustration). Sketch rate vs time. How many gallons accumulated? Show with areas.

5. Explain in one paragraph why "I'm eighteen so I take calculus" is not a readiness rule. What would you check instead?

If it isn't clicking

Diagnostic 1: Power-rule-first; no rate story.
Next move: Put rules on the shelf for three sessions. Only tables, secants, and units. Reintroduce one rule as a shortcut *after* they estimate a derivative by hand.

Diagnostic 2: Units and accumulation confused.
Next move: One day only on "rate $\times$ time = total." Use speed $\times$ hours = miles; fill-rate $\times$ minutes = gallons (composite stories). Shade area; label units on the axes and on the region.

Diagnostic 3: Birthday pressure or AP panic.
Next move: Honest readiness check — functions, trig as needed, structure, changing-quantity talk. If soft, name Precalculus or Statistics as the strong next move. If ready but anxious, shrink the week's scope to limits + rate only.

Slow down when they cannot compute an average rate with units. **Go ahead** when they estimate a derivative from a table, interpret an accumulation area, and state FTC in words — then add procedures. **Tutor** when fear of calculus has blocked attempts for weeks despite precalculus fluency — a human who teaches rate first can restart the story.

Correct without crushing: "The rule gave a function. What does the number mean at this t — and in what units?"

NMAP Finding 15 still holds: developmental appropriateness depends on prior opportunities to learn. Finding 30: faster learners may move faster when skill and motivation warrant — not as a birthday trophy.

Connection back to readiness

If Chapter 4's unit-circle and parent-function work is soft, calculus meaning will feel like fog. Return to covariation and function fluency without apology. If Algebra II structure is soft, repair it. Critical Foundations continuity still applies when unfinished number sense blocks algebra — birthday did not finish it.

A fifteen-year-old ready for limits may begin with supervision. An eighteen-year-old missing linear/quadratic structure does Algebra I repair without apology — then climbs. Skill and motivation, not a race.

More diagnostic texture

If they confuse the derivative function f' with a single number, have them build a table of estimated slopes at several x and plot those slopes — that plot *is* a rough f'. If they treat FTC as two unrelated recipes (derivative rules vs integral rules), put one prompt on the table: given f, build F(x) = _a^x f, estimate F' from the accumulation graph, and compare to f. The relationship is the point.

If parental anxiety is driving AP enrollment before readiness, name it kindly once: "We can take a strong Precalculus or Statistics year and still keep STEM doors open." Then choose the honest course.

Tools, including AI

Point back to the math hour AI box. Keep this short.

After the student has tried: graphing calculator or Desmos to check a derivative graph or to explore a parameter — labelled, after a hand attempt. Numerical derivative or integral features are checks, not replacements for table and Riemann thinking. Wolfram|Alpha or CAS to verify an antiderivative after they have a reason — not to write the free-response.

Unaided first. SCRIPT after the try for what *you* will say. No photo-to-key. No unsupervised chatbot as only partner during the attempt. Ages 15–16: co-hold in the room. Ages 17–18: more agency, still not a secret friend during the try — and not a licence for live-exam AI.

One on-age caution, then teach: in a high-school math field experiment (grades 9–11, nearly 1,000 students, Turkey), an unguarded chatbot made practice look better and left students worse on the test with the window closed; a guarded tutor that withheld the answer left the unaided test about the same as no AI (Bastani et al., 2025). That is this age band's math trial — not a U.S. homeschool RCT. Unguarded AI can inflate calculus practice and harm unaided checks; keep exit tickets closed-book.[63]

[63]Hamsa Bastani et al., "Generative AI without guardrails can harm learning: Evidence from

There is no kitchen-table trial proving CED week-by-week raises AP scores. Use the map; require meaning; keep unaided checks.

Parent script bank (say these)

- "Show me a secant, then a closer one."
- "Circle the units before you circle the answer."
- "Is this a rate or a total?"
- "What is accumulating in this shaded region?"
- "Say FTC in words — then compute."
- "You may change your mind when the table gets denser."

Copy two onto a sticky for the week. You are not a deficient parent if you use a sticky. You are prepared.

When calculus is not this year

Choose one honest path and write it down: finish Precalculus with trig graphs and covariation; or Algebra II repair plus Statistics inference talk; or a dual-enrollment college algebra/precalc if gates allow. Revisit calculus readiness in six months. Finding 15 refuses the myth that age alone finishes the prerequisites. Finding 30 allows faster movement when skill and motivation warrant — for the ready, not for the candle count.[64]

What "done enough" looks like

Calculus done-enough is for **ready** students — not every 18th birthday.

- ☐ Explains derivative as instantaneous rate with units on a graph or table prompt.

high school mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. ON-AGE high-school math field experiment (Turkish grades 9–11); not a U.S. homeschool RCT. Unguarded AI can inflate practice and harm unaided checks — keep exit tickets closed-book.

[64]National Mathematics Advisory Panel, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (U.S. Department of Education, 2008), Finding 15: place by skill; age-stage "not ready" claims are wrong when prerequisites are present — and birthday alone does not create them. Finding 30: acceleration when skill and motivation warrant — for the ready, not for the candle count. Calculus is for ready students — not every eighteenth birthday.

- ☐ Computes average rates on shrinking intervals and estimates a derivative numerically.
- ☐ Computes routine derivatives taught in the course (power, sum, and others you have introduced) *after* rate meaning.
- ☐ Explains integral as accumulation / net change; approximates area with sums.
- ☐ States FTC in words linking accumulation and rate; uses it in at least one straightforward computation.
- ☐ Uses graph, table, and equation on one prompt (multirepresentational grain — College Board map labelled).
- ☐ Limits: estimates from table/graph; connects continuity talk in plain language.
- ☐ Notebook: dated exits; one in-home and one out-of-home rate or accumulation try-it.
- ☐ Does *not* treat birthday as the readiness gate; does *not* treat BC completion as required virtue.

If the year's work was Calculus, the stranger-readable title is **Calculus**. Never a brand. Never Emerging Adults Math I. AP score is optional evidence, not the only path that counts.[65]

If done-enough fails, return to Precalculus covariation and function fluency, or take a Statistics year with head high. Revisit Tuesday's rate table on Thursday. Mix a limit with a units prompt. Stop when the exit ticket is honest. That rhythm lives in the math hour, this checklist, and records — not in a ninth teaching chapter.

Many strong students never take high-school calculus and still thrive in college STEM after a later start. Many take calculus with meaning and never sit an AP exam. Both can be right. Place by skill.

A student who finishes a rate week able to speak units and estimate a derivative from a table has learned something real — even before an AP practice free-response. A student who finishes an accumulation week able to shade a region and name gallons or miles has learned something real — even before a fancy antiderivative. Stack those weeks. Procedures will have a place to live.

[65] Ellena Sempeles and Jiashan Cui, *Parent and Family Involvement in Education: 2023*, NCES 2024-113: no national homeschool diploma. Transcript title **Calculus** only by content — never a brand, never "Emerging Adults Math I." An AP score is optional evidence, not the only path that counts.

If you only remember one teaching move from this chapter: when you hear a naked rule, ask what the average rate is becoming and in what units. Then wait. Then point at the graph or the table. That move alone turns calculus hour from flashcard theater into mathematics.

Composite illustrations labelled

A speedometer story with made-up speeds is fine when labelled as an illustration. A tank-filling story is fine the same way. Sports splits from a real public box score are fine as ordinary public data. Skip inventing a study about teens and claiming it was measured. Skip portraits of real adolescents. Still-lifes of graphs and tables carry the chapter's pictures.

This is a useful study orientation (Thompson line; CED as map; Tallman & Carlson sobriety on procedure-heavy exams), not a promise that every home will see the same result. Then teach — tables, secants, units, accumulation, FTC words, procedures after.

Keep the pencil in their hand. Your job is the question, the wait, and the units. Their job is the secant, the estimate, and the sentence about accumulation. That division of labor is the math hour applied to calculus.

Chapter 6 — Statistics and inference talk

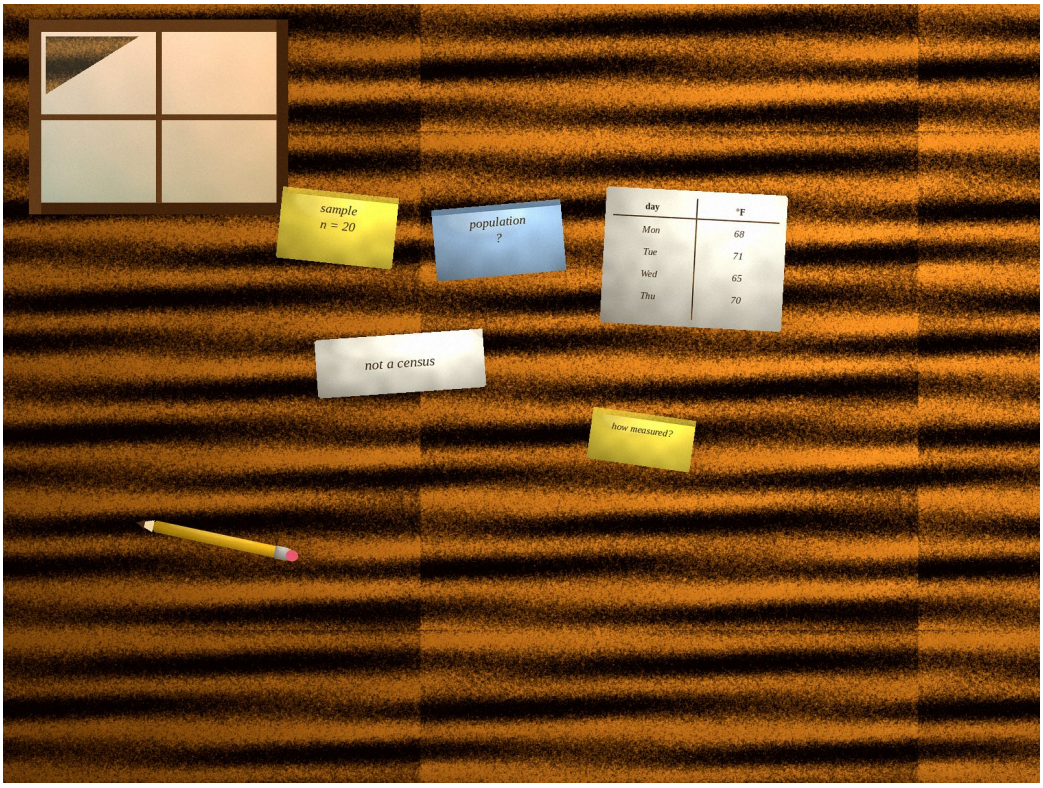

Figure 9: A kitchen-table still-life: two sticky notes labelled SAMPLE and POPULATION; a tiny dot plot; a co-op survey scrap with ordinary stakes. No people. No logos.

Why this matters

Inference talk is mathematics for emerging adults: careful claims, ordinary data, and the humility to say what a sample cannot prove. It belongs beside algebra and geometry — not as an afterthought and not as a culture-war workbook.

A number about twenty teenagers is not automatically a claim about all teenagers.

That sentence is the heart of this chapter. Sample versus population. Ordinary stakes. Inference *talk* for every reader; a full inference course (confidence intervals, significance tests, AP Statistics depth) only when the course skill calls for it — not required every week for every junior.

At 15–16: displays, summary measures with meaning, awareness of bivariate association, basic probability, and always "how was this measured?" At 17–18: sampling variability as an idea; informal toward formal inference for ready students; regression meaning; simulation. Refuse "calculator button = understanding." Refuse an AP Statistics score as the only statistics that counts. Refuse live political controversy as "data literacy." Ordinary stakes only: a co-op survey, a sports sample, a week of temperatures, board-game scores, a museum display.

Schemas for story problems matter more than keyword hunts. "How were these data produced?" beats "which formula matches the word *average*." A mean that hides an extreme day is a teaching object. A sample proportion offered as a census is a teaching object. You can teach both at the kitchen table without a culture-war dataset.

Why struggle with careful language?

Because careless language is how a sample of twenty becomes a myth about millions. Mathematics at 15–18 includes the discipline of saying what you know and what you do not. That discipline helps in science labs, in reading charts online, in dual-enrollment research methods later, and in ordinary citizenship — without turning this chapter into a civics war. Keep the dataset boring. Keep the standards on the sentence high.

CCSS-M Statistics and Probability categories can sit in the background as a topic map. NCTM's statistics domains (quantitative literacy; visualizing/summarizing; inference; probability) name the same neighborhood. Neither document is a statute in your kitchen. Both support the claim that high-school mathematics includes

stats talk — not only algebra and geometry.

Parent learns this week: hear a sample mean offered as a population claim, and ask which it is. Student tries today: run a tiny sample, state the sample statistic, and write one sentence about what would be needed to claim something about a population.

For the parent: understand it yourself

You do not need to be a statistician. You do need to hear a sample mean sold as a population fact, a button press with no interpretation, a "30% chance" treated as a promise about your block, and a request for a culture-war data prompt — and to redirect to ordinary stakes without taking the pencil.

One everyday picture

You ask twenty students in a co-op whether they prefer morning or afternoon classes. Twelve say morning. In *this sample*, 12/20 = 60% prefer morning. That 60% is a sample statistic. The population might be "all students in this co-op," or "all homeschooled teens in our county," or something larger — and you have not measured that population. To claim about the population you need a clearer population definition, a sampling plan that is not only whoever showed up on Tuesday, and honest language about uncertainty. The kitchen move is to say: "In this sample of 20, 60% preferred morning. We have not shown that 60% of all American teens prefer morning."

One precise picture

Population: the whole group you want to talk about.
Sample: the part you actually measured.
Statistic: a number computed from the sample (sample mean $\bar{x}$, sample proportion $\hat{p}$).
Parameter: the corresponding number for the whole population (often unknown).

Displays (dot plot, histogram, boxplot, scatterplot) show shape, center, spread, and oddballs. Summary measures: mean and median as center; range, IQR, standard deviation as spread talk — start with "what the average hides" before you

demand formula fluency. Probability: chance models for coins, spinners, and simple games; a probability is not a promise that your next week will obey the long-run fraction. Association: two variables may move together on a scatterplot; association is not automatic proof of a simple cause story — keep stakes ordinary and claims modest.

Inference talk (every reader): samples vary; bigger samples usually reduce chance error (qualitative); biased sampling breaks claims no matter the size; say what you measured. Formal inference (optional depth): confidence intervals and significance tests when the course is AP Statistics or a dedicated unit — College Board's AP Statistics CED (labelled College Board map) organizes explore data; sampling and experimentation; probability and simulation; inference; and practices Formulate Questions / Collect Data / Analyze Data / Interpret Results, with prerequisite first-year algebra. Label it a map. Full AP depth is not required for every reader of this book.[66]

NCTM Catalyzing Change names quantitative literacy, visualizing and summarizing, inference, and probability among statistics domains for high school — useful headings, not a kitchen statute.[67] CCSS-M Statistics and Probability is another map.[68] NAEP Grade 12 results describe a school population; they are sobriety about national school performance, not a homeschool census and not your child's score.[69]

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[66]College Board, *AP Statistics Course and Exam Description* (effective Fall 2026): explore data; sampling and experimentation; probability and simulation; inference; practices Formulate Questions / Collect Data / Analyze Data / Interpret Results; prerequisite first-year algebra. Labelled College Board map — not a kitchen RCT. Full AP Statistics depth is optional, not required of every reader of this book. Access date for URLs in these notes: 5 September 2026.

[67]National Council of Teachers of Mathematics, *Catalyzing Change in High School Mathematics* (Reston, VA: NCTM, 2018), statistics domains among Essential Concepts (executive-summary grain): quantitative literacy; visualizing and summarizing; inference; probability — useful headings, not a kitchen statute.

[68]Common Core State Standards for Mathematics (2010), Statistics and Probability conceptual category — a map of typical U.S. public-school placement, not a homeschool statute. http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf.

[69]U.S. Department of Education, National Center for Education Statistics / National Assessment Governing Board, 2024 NAEP Mathematics Grade 12 national results: school population (average about 147; about 22 percent at or above Proficient; about 45 percent below Basic) — sobriety about national school performance, not a homeschool census and not your child's score. NCES 2024221 pattern as opened 5 September 2026.

Wrong answers you should be able to hear

1. **Sample mean offered as population claim.** "Teens sleep 6.2 hours" after averaging twelve co-op answers. Ask: "Is this number about the sample, or a claim about the population?"
2. **Button-pushing without interpret.** They produce $\bar{x}$ or a regression line from a calculator and cannot say what it means in the context. Ask for a sentence with the units and the group measured.
3. **"30% chance" as a promise about the block.** A model says rain probability 30%; they treat it as a guarantee about their street. Ask what a probability claims — and what it does not.
4. **Treats a mean as the whole story.** One average, no spread, no extreme day. Ask what the average hides; show the list.
5. **Wants a culture-war data prompt.** Redirect to ordinary stakes: sports, weather, games, co-op survey, exhibit data. This book does not rehearse live fights as content.
6. **Dumps AP Stats as required for every junior.** Name inference talk as the object for everyone; full AP depth as optional course skill.
7. **Cannot say how the data were produced.** They analyze a table with no idea whether it was a voluntary poll, a census of one classroom, or a random sample. Ask for the production story first.

Five-minute parent warm-up

List five board-game scores or five temperatures from a weather app week. Compute the mean by hand. Circle the most extreme value. Say out loud one thing the mean hides. Then write two sentences: one that correctly describes the sample, and one overclaim about a larger population. Cross out the overclaim. That is today's ear training.

Displays and summaries — enough to teach well

You do not need every plot type in week one. A sorted list plus a dot plot teaches shape and oddballs. A histogram bins continuous data when lists get long. A boxplot (when the course introduces quartiles) compresses center and spread. Scatter-

plots show two quantitative variables together; describe form, direction, strength in plain words before fitting a line.

Mean vs median: the mean uses every value and moves with extremes; the median resists a single wild point. Ask which is more honest for *this* list. Standard deviation can wait until spread talk in words is easy — "how far do points tend to sit from the center?" — then introduce the formula your course uses. Refusing the button until the sentence exists is the habit.

Probability without superstition

A fair coin does not "owe" you heads. Independence means one flip does not change the next in the model. Simulation builds intuition: many trials, display the results, talk about long-run relative frequency. Conditional language ("given this, what's the chance of that?") waits until two-way tables appear — still ordinary stakes (game tokens, class survey topics), never a culture-war contingency table as the spine.

Formal inference — optional depth map

When the year's course *is* Statistics or AP Statistics, you will teach confidence intervals as "plausible values for a parameter given sample data and a method," and significance tests as "is this sample surprising under a null claim?" Guardrails: conditions matter; "significant" is not "important" in everyday English; a p-value is not the probability the null is true in the simple way students wish. Keep claims modest. This book does not require every reader to finish that arc.

College Board's AP Statistics CED is a labelled College Board map (explore; sample/experiment; probability/simulation; inference). Prerequisite talk: first-year algebra. Use it if you are AP-aligned. Do not treat it as a home RCT.

How to teach it this week

This week's plan: one focus — sample vs population language *or* center/spread with meaning *or* a simple probability model. Four or five sessions. One in-home and one out-of-home try-it. Friday: "sample or population?" plus one design question on the exit ticket.

Session shape

Typical session: 45–75 minutes.

1. **Warm-up (5 min).** Mean from a tiny list; ask what it hides. Or: identify whether a headline sentence is sample or population language.
2. **Short model (8–12 min).** Sample vs population on ordinary stakes (co-op survey composite). Write one true sentence and one overclaim; cross out the overclaim together.
3. **Student attempt (15–20 min).** Tiny sample (coin, spinner, or survey of household/co-op with permission); compute a statistic; write the careful sentence.
4. **One good question, then wait.** Locked opening below.
5. **Mixed practice (10–15 min).** One display read, one mean/median compare, one sample-vs-population edit, one "how measured?" prompt. Mix a prior algebra skill if energy allows.
6. **Exit ticket (5 min).** "Sample or population?" + one design question (who was measured? who was not?).

Exact wording you can say:

- "Is this number about the sample, or a claim about the population?"
- "How was this measured — or sampled?"
- "What does this average hide?"
- "What word should we use so we don't overclaim?"
- "You may change your mind when we look at how the data were produced."

How to fade help: Day 1 you write the careful sentence template with them. Day 2 they fill the blanks. Day 3 they write both the true sentence and the overclaim, then strike the overclaim alone. When to stop talking: after the locked question. Wait. Look at the sticky notes or the table, not at their face.

Sample week

- **Monday:** Sample vs population stickies; rewrite three sentences.
- **Tuesday:** Week-of-temperatures or game scores; mean vs extreme day; spread talk.
- **Wednesday:** Sports box score out-of-home *or* museum/library exhibit data.
- **Thursday:** Coin/spinner sample of 20; sample proportion; what would you need to claim about a "true" proportion?

- **Friday:** Exit ticket; done-enough; mix any prior skill.

Ages 15–16

Stay with displays, summary measures with meaning, basic probability, and ruthless sample-vs-population language. Scatterplots and "association" awareness without forcing causal sermons. Schemas for story problems: what was measured, on whom, how chosen, what question. Keyword lists ("the word *average* means mean") are not the spine.

Ages 17–18

Add sampling variability ideas; simulation (dice, coins, technology after attempt); regression meaning on ordinary bivariate data; informal to formal inference if the course is Statistics or AP Statistics. Still ordinary stakes. Still refuse live controversy as the dataset. Full AP depth remains optional — appropriate when that is the year's course skill, not as a universal junior requirement.

A longer model you can reuse

Write this on the board or a sheet once, then reuse all year:

Population we care about: _____
Sample we actually have: _____
How chosen: _____
Statistic: _____
Careful claim: _____
Overclaim we refuse: _____

Run a sports box-score number through the template. Run a co-op poll through it. Run a museum chart through it. The template is the curriculum object; the contexts change.

Association and regression (17–18 grain)

When two quantitative variables appear together, students should:

1. Plot first.

2. Describe form (linear/curved/clustered), direction (positive/negative), strength (tight/loose) in words.
3. Only then consider a line or r if the course includes them.
4. Interpret slope in context with units ("about ___ more y per additional x") as a sample description — not as destiny for every individual.
5. Keep residual talk light: points miss the line; that miss is information.

Refuse causal sermons from a single observational scatterplot. Ordinary example: hours practiced vs free-throw percent for one team's week — association talk, not a TED talk.

Designing a tiny study (schema)

Question → population of interest → sample plan → data collection ethics (consent, ordinary topics) → display → statistic → careful sentence → limitations. That sequence *is* AP Stats practice Formulate/Collect/Analyze/Interpret at kitchen scale. You can run it without ever printing "AP."

Mixed practice ideas

- Edit three headlines into careful sample sentences.
- Compare mean and median on a list with an outlier.
- Coin sample vs spinner sample — two proportions.
- Read a museum chart; write one read-beyond caution.
- Revisit an Algebra I percent problem and ask sample vs population if the percent came from a survey story.

First try-it for the student (in-home)

Coin / spinner sample of 20

- **Time:** 10–15 minutes.
- **Materials:** fair-ish coin or spinner; notebook.
- **Safety:** none special.
- **The fun:** Record 20 spins or flips; compute sample proportion of "heads" or a chosen color; argue what you would need to claim about a long-run or population proportion.
- **The skill:** Language of uncertainty; sample vs population.

Schemas before keywords — expand

Story problems in statistics fail when students hunt the word *average* or *probability* and grab a formula. Teach a repeated schema (also in the practice section): measured trait → group → selection method → sample or population claim → careful sentence → what center hides. Write the schema on a card. Use it until it is boring — boring means fluent.

For experiments vs observational stories at 17–18: experiments assign treatments; observational studies watch. Causal claims from observational data need caution. Keep examples ordinary: plant fertilizer on two trays (experiment); comparing sleep and quiz scores from a voluntary form (observational — association talk, not a morality play).

Guardrails for this chapter

There is no sports-stats trial this book leans on as a kitchen guarantee. Philosophy-for-Children findings are not math findings — keep them out of chance week. NAEP Grade 12 describes a school population, not a homeschool census. Full AP Stats depth is optional, not required of every reader. Live political, religious, or identity controversy stays out of the dataset. No public quiz of a cashier. No scraping private data. Ordinary stakes only.

Practice that actually builds learning

Sports box scores and exhibit displays motivate. They do not replace the careful sentence.

Talk box

Opening question (locked): "Is this number about the sample, or a claim about the population?"

Follow-ups:

1. **Meaning:** How was this measured or sampled? Who is missing?
2. **Representation:** Show me on the table, sticky notes, or display — circle the sample.

3. **Type / check:** What does this average hide? Point to an extreme or to the spread.
4. **Chance language:** Does this chance number promise your block or your next week? What does it claim instead?
5. **Revise:** What word should we use so we don't overclaim? You may change your mind.

How to wait: Ask, slow three, wait again after they pause. Look at the data.

What a stuck silence usually means: Button-pushing habit; treating sample mean as census; wait-time 1 was zero; they want a culture-war prompt (refuse); they cannot say how data were produced; they think you want a vocabulary word instead of a reason. Point back at sample vs population stickies. Do not fill silence with the "right" civic speech.

Named try-its

1. Coin / spinner sample of 20 (in-home) — as above. Extend: two samples of 20; notice they disagree; that disagreement *is* sampling variability talk.

2. Week of temperatures / board-game scores (in-home)

- **Time:** about 5 minutes/day + Friday summary.
- **Materials:** weather app highs or family game scores; notebook.
- **Safety:** none special.
- **The fun:** Friday mean vs one extreme day; sketch a quick display.
- **The skill:** Center; spread talk; what the average hides.

3. Sports box score (out-of-home)

- **Time:** during a game or while reading a public box score afterward.
- **Materials:** notebook; publicly posted stats.
- **Safety:** ordinary event courtesy. No harassment. Sports stats are rates — not identity fights.
- **The fun:** Batting average, shooting percent, or yards per carry as a ratio; what the number claims about this player in these games — and what it does not claim about all players everywhere.
- **The skill:** Sample-ish sports numbers vs overclaim; ratio as statistic.

4. Library / museum exhibit data (out-of-home)

- **Time:** 10–15 minutes.

- **Materials:** notebook; an exhibit with a chart or table (science museum, history display, library poster).
- **Safety:** ordinary public manners. Do not photograph people without consent; data displays are fine when allowed.
- **The fun:** Read-the-data / read-between / read-beyond: what does it say, what might it imply, what can you not conclude?
- **The skill:** Question the display; ordinary stakes.

5. Co-op survey talk (composite, in-home)

- **Time:** 10 minutes.
- **Materials:** sticky notes; a tiny voluntary poll of household or co-op peers *with permission*.
- **Safety:** consent; no pressure; no shaming answers; keep topics ordinary (morning vs afternoon classes, favorite board game — not politics, religion, or identity fights).
- **The fun:** "60% of 25 teens prefer morning classes — *in this sample*." Practice refusing the leap to "all American teens."
- **The skill:** Overclaim detection; sample sentence writing.

Schemas for story problems (teach these moves)

When a word problem or article appears, walk a schema — not a keyword list:

1. **What was measured?** (hours of sleep, points per game, temperature)
2. **On whom / on what?** (which people, which games, which days)
3. **How chosen?** (everyone in the room, volunteers, random draw, unknown)
4. **Sample or population claim?** (circle which)
5. **What would a careful sentence say?**
6. **What does a center number hide?** (spread, extremes, subgroups)

Keyword grab ("I saw *average* so I divide") is the incorrect example pattern to diagnose. Schema first.

Blocked practice

New move: only rewrite overclaims into sample-honest sentences for a day. Or only "what does the mean hide?" with tiny lists.

Mixed practice

When careful sentences come easily, mix: display reading, mean vs median, probability model, sample-vs-population edit, one prior algebra or function item.

One incorrect example to diagnose

Headline-style sentence in the notebook: "Homeschool teens prefer morning classes — 60%." Source: 25 co-op volunteers on one Tuesday.

What it usually means: sample offered as population; volunteer bias ignored. Ask the locked question. Have them rewrite: "In a voluntary sample of 25 teens at one co-op on one day, 60% said they prefer morning classes." Ask what would be needed for a broader claim.

Ages band detail — worked session snippets

15–16 snippet (25 minutes inside a longer hour): Warm-up mean of five game scores. Model: sticky SAMPLE vs POPULATION. Attempt: rewrite two overclaims. Talk box locked opening once. Exit: one careful sentence. Done. Tomorrow add a display.

17–18 snippet (full hour): Warm-up: identify bias in a voluntary poll story. Model: sampling variability with two samples of 20 from a spinner. Attempt: students run their own two samples; compare proportions. Question: locked opening about whether their proportion is the spinner's "true" value. Mixed: scatterplot description in words. Exit: sample-or-population plus "what would a larger sample do to chance error?" (qualitative).

Incorrect examples bank (diagnose, don't shame)

1. "The probability of rain is 40%, so it won't rain on our block." — Chance treated as local promise. Redirect to model language.
2. "r = 0.9 so hours of practice *cause* higher scores." — Association sold as cause from one observational set. Ask for plot + caution.
3. "n = 1,000 so the survey is fine" when the thousand are volunteers from one website. — Size without design. Ask how chosen.
4. Calculator outputs $\bar{x} = 72.384762$ and the student copies all digits as "more accurate." — False precision. Round sensibly; interpret.

5. "AP Stats is the only real stats." — Refuse; name inference talk as real; AP as optional depth.

Out-of-home safety and courtesy

Museum and library: walk, look, write; do not block exhibits; follow photo rules. Sports: spectator math, not coaching unsolicited. Store percent try-its if you connect to Chapter 1/3 money math: aisle courtesy, no shoplifting a "test," no cashier quiz. Co-op surveys: consent, ordinary topics, opt-out without shame.

Materials fit (not rank)

Kitchen notebook first. Open data from weather apps and public box scores. Khan or OpenStats-style practice after attempts if you use them. If you adopt a full AP Statistics text or video course, title the year **Statistics** when content matches. WWC school ratings for algebra software are not a crown for a statistics brand, and this book does not supply a secondary stats WWC win.[70] Fit your load and the student's readiness.

For the student

Statistics begins with a careful sentence.

If you measured part of a group, your number is about that part — the sample — until you have a honest plan and honest language for talking about the whole — the population.

Tiny worked example

You flip a coin 20 times and get 12 heads. Sample proportion: $\hat{p} = 12/20 = 0.6$.

Careful sentence: "In this sample of 20 flips, 60% were heads."

[70] What Works Clearinghouse Secondary Mathematics intervention reports (e.g., Saxon Algebra I, May 2016; Cognitive Tutor; UCSMP Algebra; Core-Plus) are school-side ratings. They are not a crown for a statistics brand at the kitchen table, and this book does not supply a secondary statistics WWC effectiveness win.

Overclaim: “This coin is 60% heads forever” or “All coins land heads 60% of the time.”

What would you need for a stronger claim about the coin’s long-run behavior? More flips, a clearer model, and still language about uncertainty — not a promise about the next flip.

Two tries

1. List eight scores from a game or assignment. Compute the mean. Which value pulls it? Write one sentence about what the mean hides.
2. Find a chart in a museum, library, news-lite weather page, or textbook. Answer: What does it say? What might someone wrongly conclude? Sample or population?

Explain it back

Explain sample vs population to a parent using one sticky note for each word. Give one example sentence that stays honest.

Challenge

Design a five-question ordinary-stakes survey for a co-op or household (permission required). Predict bias sources before you run it. Run it with n at least 10 if you can. Report only sample claims. Write one paragraph: what you can say, and what you refuse to say.

Optional (17–18 / ready): simulate 20 sample proportions of size 20 from a spinner; display them; talk about how samples vary.

You are allowed to wait. You are allowed to change your mind when you learn how the data were produced. You are allowed to take a full AP Statistics course later — or to stop at strong inference talk plus Algebra II / Precalculus. Both can be enough.

More tries

3. Rewrite this overclaim into a careful sentence: “Our co-op’s quiz average is 82, so American teens average 82 on this quiz.” What information is

missing?

4. Create a dot plot of ten numbers you choose. Move one number far away. What happens to the mean? To the median?
5. A spinner is one-fourth red. Simulate 40 spins (paper clips and a pencil work). Compare your sample proportion of red to 0.25. Did you "prove" the spinner is unfair? Why or why not?

How to teach it — fading help and wait time

Day 1 you write the careful sentence with them. Day 2 they fill blanks in a template. Day 3 they write true claim and overclaim alone. Day 4 they teach a sibling or parent using the stickies. When silence comes after the locked question, wait a slow three (classroom think-time convention — not a math RCT). Wait again after they pause. Look at the table or stickies. Stuck silence often means button habit, census-thinking, zero wait-time, or a wish for controversial data — refuse the last, fix the others by pointing at production of the data.

Blocked then mixed — stats version

Blocked: ten minutes of only rewriting overclaims. Or only mean-vs-extreme on short lists. Mixed (when ready): one display, one probability simulation sentence, one sample-vs-population edit, one schema walk on a short article paragraph about sports or weather (ordinary). Revisit Tuesday on Thursday. That is the habit.

Student challenge extension

If the five-question survey challenge is too much this week, shrink it: three questions, household only, n 5, still report only sample claims and one bias source. Completing a tiny honest study beats designing a grand poll that never runs.

Records pointer

File one corrected careful paragraph and one display in the year's folder if you title the year Statistics. If stats talk lived inside Algebra II, keep those pages inside the Algebra II notebook and do not print a Statistics credit unless the year's work truly maps. Stranger-readable titles only by content. Dual enrollment / AP / CLEP remain options with caveats in the records chapter — not this chapter's spine.

If it isn't clicking

Diagnostic 1: Sample sold as population every time.
Next move: Sticky notes on the wall for a week — every statistic gets labelled SAMPLE or POPULATION CLAIM. No unlabeled numbers in the notebook.

Diagnostic 2: Calculator or spreadsheet first; no sentence.
Next move: Hand computation of mean on lists of length ≤ 10; sentence before device. Device returns only after the sentence exists.

Diagnostic 3: Wants controversial data to feel "real."
Next move: Refuse. Offer sports, weather, games, exhibit math. Explain once: this book teaches mathematics with ordinary stakes; live fights are not the worksheet. Then teach.

Slow down when they cannot write one careful sample sentence. **Go ahead** when they edit overclaims, read a display skeptically, and say how data were produced. **Tutor or dedicated Stats course** when they are ready for formal intervals and tests — optional depth, not a shame gate.

Correct without crushing: "Your arithmetic is fine. The claim stepped past the sample — let's rewrite the sentence."

Placement and titles

If the year's work is a dedicated statistics course, title it **Statistics**. If stats talk is embedded inside Algebra II, keep the transcript honest — do not print Statistics unless the year's work maps. Course brand names stay in the materials footnote. Never Emerging Adults Math I.

Re-placement midyear is allowed. A student who thrives on simulation and inference may shift toward a Stats-heavy spring. A student who needs more algebra structure should repair algebra — statistics will still be there.

More diagnostic texture

If every probability is treated as a prediction about tomorrow morning, return to long-run language and simulation displays. If regression is a button that spits r and a line with no look at the scatterplot, ban the button for a week; require a hand description of form/direction/strength first. If they refuse to write sentences

because "math is numbers," make the exit ticket *only* sentences for three days — then return to computation.

Tools, including AI

Point back to the math hour AI box.

After the student has tried: calculator or spreadsheet to check a mean or make a quick graph — labelled, after hand work on small lists. Simulation tools (calculator random integers, spreadsheet `RAND`, classroom software) after they understand one hand simulation. Graphing calculators' Stats menus are fine for larger data *after* interpretation habits exist.

Do not photo-to-key a stats worksheet. Do not let a chatbot invent a fake study or a controversial dataset as "practice." Do not detector-grade their write-ups. Ages 15–16: co-hold in the room. Ages 17–18: more agency, still not a secret friend during the attempt.

One on-age caution, then teach: in a high-school math field experiment (grades 9–11, nearly 1,000 students, Turkey), an unguarded chatbot made practice look better and left students worse on the test with the window closed; a guarded tutor that withheld the answer left the unaided test about the same as no AI (Bastani et al., 2025). That is this age band's math trial — not a U.S. homeschool RCT. The model may prepare you; it may not write their interpretation paragraph during the try.[71]

Parent script bank (say these)

- "Sample or population?"
- "Who was measured — and who was not?"
- "What does the average hide?"
- "How were these data produced?"
- "What word keeps us from overclaiming?"
- "You may change your mind when we learn the sampling story."

[71] Hamsa Bastani et al., "Generative AI without guardrails can harm learning: Evidence from high school mathematics," *Proceedings of the National Academy of Sciences* 122, no. 26 (2025): e2422633122. ON-AGE for this book — not a U.S. homeschool RCT. The model may prepare the adult; it may not write the student's interpretation paragraph during the try.

Life-of-the-habit without a ninth chapter

Revisit Tuesday. Mix types on Thursday. Keep the dated notebook. Stop when the exit ticket is honest. File one corrected careful paragraph for records. That is the habit. It does not need its own teaching chapter title.

What “done enough” looks like

Inference talk done-enough for every reader; full AP Stats done-enough only if that was the course:

- ☐ Labels a number as sample statistic or population claim in plain language.
- ☐ Writes one careful sample sentence and rejects one overclaim.
- ☐ Computes and interprets a mean (and median when useful) on a small list; says what a center number can hide.
- ☐ Reads a simple display (table, bar/dot/histogram, or scatterplot) with a read-the-data / read-beyond habit.
- ☐ Asks how data were produced before trusting a claim.
- ☐ Uses a basic probability model (coin/spinner) without treating chance as a promise about one block.
- ☐ (17–18 / course-dependent) Describes sampling variability qualitatively; optional: informal or formal inference moves taught in-course.
- ☐ (Optional AP depth) Meets your chosen course’s exit skills for intervals/tests/experiments — not required of every reader of this book.
- ☐ Notebook: dated exits; coin/spinner or survey try-it; one out-of-home data read.
- ☐ No live-controversy datasets as the spine of practice.

If the year’s work was Statistics, the stranger-readable title is **Statistics**. Never a brand. Never Emerging Adults Math I. An AP Statistics score is optional evidence, not the only path that counts.[72]

Life-of-the-habit tips live here and in records — not in a ninth teaching chapter: revisit Tuesday’s careful sentence on Thursday; mix display reading with sample-vs-population edits; keep a dated notebook; stop when the exit ticket is honest. A

[72] Ellena Sempeles and Jiashan Cui, *Parent and Family Involvement in Education: 2023*, NCES 2024-113: no national homeschool diploma. Transcript title **Statistics** only by content — never a brand, never “Emerging Adults Math I.” An AP Statistics score is optional evidence, not the only path that counts.

diploma is not this chapter's promise. A percentile is not. A path through high-school mathematics — including honest statistics talk — is.

Place by skill. A student who finishes Algebra II with strong inference talk and no AP Stats can be done enough for this chapter's core. A student who wants formal inference can go further. Both are respectable. The aim is judgment with numbers — the humility to measure twice before you speak about a whole population.

If you only remember one teaching move from this chapter: when you hear a number about a group, ask whether it is about the sample or a claim about the population — then wait — then ask how the data were produced. That move alone turns statistics hour from button theater into judgment.

Revisit Tuesday's careful sentence on Thursday. Mix a display with a probability model. Keep the notebook dated. Stop when the exit ticket is honest. Ordinary stakes. Real mathematics. Place by skill.

This is a useful study orientation for stats talk (sample vs population; ordinary stakes; optional AP depth), not a promise that every home will see the same result. Then teach — stickies, careful sentences, tiny samples, displays, and schemas before keywords.

One more in-home variant

Two sticky notes, every article: When anyone in the house quotes a percent from a podcast or a box score, pause for thirty seconds. Sticky one: SAMPLE. Sticky two: POPULATION CLAIM? Sort the quote. If you cannot tell how the data were produced, that *is* the finding — unknown production, weak claim. Thirty seconds of sorting beats a lecture. Do this three times in a week and the ear trains fast.

One more out-of-home variant

Park or trailhead sign data: Many parks post trail lengths, elevation gains, or visitor counts. Copy one chart into the notebook. Ask the locked question. Ask what a visitor count for one weekend does not claim about a year. Ordinary outdoor math; no climbing; stay on marked paths.

Keep the pencil in their hand. Your job is the question, the wait, and the refusal of overclaim. Their job is the sample sentence and the production story. That division of labor is the math hour applied to statistics.

Friday habit for this chapter: one overclaim autopsy (rewrite it) plus the done-enough checklist items you own this week. Date the page. If the student is in a full Statistics course, add one procedural check your materials require — still after the careful sentence, never instead of it.

Chapter 7

Records

The math *work* can be as serious as any kitchen's. The *credit* is a family claim until a stranger can use it — a college admissions reader, an NCAA Eligibility Center evaluator, a dual-enrollment office, a receiving school's counselor, or the high-school transcript you will someday sign. That stranger will not have sat at your table. They will have a page.

This chapter is how you make that page at ages 15–18. Lawful, honest, and readable to a stranger are three jobs. Keep them in view.

This week, date one free-response page a stranger could almost read. Today the student tries it; you write the line.

Figure 10: An open notebook, a dated free-response with a correction in the margin, a sticky note titled Precalculus, a pencil. No diploma seal. No brand logo as the title. No adolescent.

What you are making

You are making a **dated notebook of mathematics**, with the skill named, plus an **exit ticket**, plus **one diagnostic item**, plus optional publisher placement to start a purchased program or restart after a gap. You are not making a national diploma. You are not making Emerging Adults Math I.

When a stranger asks, the title is **Algebra I**, **Geometry**, **Algebra II**, **Precalculus**, **Calculus**, or **Statistics** — only if the year's work was that course a stranger can map. Never a brand. Never AoPS Intermediate Algebra, Saxon Advanced Math, Teaching Textbooks Algebra 2, CTCMath Grade 10, or **Emerging Adults Math I** as the credit. The dated notebook and the exit ticket live *inside* those ordinary

titles. Materials may sit in a parenthetical or a course-description line: "Algebra II — text: Foerster; dated notebook." Combining objects is allowed. Combining does not mint a new title.

There is no federal homeschool diploma and no national mathematics credit.[73] Texas, North Carolina, and New York do not issue one. Pennsylvania's supervisor or approved-organization diploma is a high-school object under that statute. Virginia's list of subjects is a list, not a transcript. A parent-issued diploma can be a real piece of paper. It is not a registrar's national Algebra I credit.

Two readers, if anyone outside the kitchen asks: the **state** (notice, IHIP, portfolio, math-inclusive test, evidence of progress) and a **later stranger** (transfer, dual enrollment, college admissions, NCAA Eligibility Center). State compliance is not a math diploma. Neither a thick portfolio nor a silent lawful year is a reason to fake a credit — or to skip the dated notebook.

A family that invents "Emerging Adults Math I, 1.0, A" for a sixteen-year-old has created a slogan. A receiving counselor cannot map it. NCAA core worksheets, Advanced Placement, and CLEP College Algebra belong to this band as *options* — they are not the spine of the year. SAT Subject Tests were discontinued — U.S. sittings ended immediately in January 2021; last international sittings were May and June 2021.[74] One sentence. Stop. They are not this year's check. Do not send a junior in 2026 to a Subject Test.

The last published federal subject map for grades 9–12 still shows **arithmetic** taught that year to 31 percent, **Algebra I** to 33 percent, **Algebra II** to 33 percent, **Geometry** to 26 percent, and **Calculus** to 13 percent.[75] That is why this book places by skill. If a thick state later wants a log: name the skill, date it, keep a sample.

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[73] No national homeschool diploma (TEA; NC DNPE; PA § 13-1327.1; NY 8 NYCRR 100.10; VA subject list is not a transcript). Access date for URLs in these notes: 5 September 2026.

[74] College Board, 19 January 2021 announcement: SAT Subject Tests discontinued; U.S. sittings ended immediately; last international administrations May and June 2021. Do not send a 2026 student to a Subject Test.

[75] NCES 2020-001, Table 9 (grades 9–12, taught that year): arithmetic 31; Algebra I 33; Algebra II 33; Geometry 26; Calculus 13. Table 8 *ever*: Algebra I 79; Geometry 52; Algebra II 49; Calculus 15! (interpret with caution). NCES 2024-113 table A-6: 3.4 percent / ~1,765,000 homeschooled; HS cell 3.1 percent. Instruction-at-home is a different bucket.

Titles a stranger can read

Print these and only these as high-school math names, if a name is needed at all:

Algebra I. Geometry. Algebra II. Precalculus. Calculus. Statistics. Reading / Language Arts. Science. Social Studies / History (if taught). **Art, Music, Physical Education.**

Those names travel. They are the names public high-school reports already use. They are the names the NCAA sample academic record already uses — Algebra I, Geometry, Algebra II, Precalculus.[76]

Never a brand. Never Emerging Adults Math I.

The brand, if used at all, sits in a parenthetical or in a materials line: "Geometry — text: Jacobs Geometry; AskDrCallahan online course; dated notebook." Or: "Precalculus — Saxon Advanced Math; lessons covering trig and analytic geometry; dated notebook." Or: "Calculus — OpenStax Calculus; dual enrollment at [Community College]; college transcript on file." Kitchen function-family cards plus library text plus one video lesson is still Algebra II.

A publisher's "grade 11 Algebra II workbook" is a scope, not a legal grade, and not a transcript line. Placement is by skill, not birthday. Algebra I before Geometry is the **typical** U.S. kitchen sequence. It is not a federal statute.[77] Integrated pathways exist as school objects; if you use one, still export stranger-readable titles or a one-page course description that maps content to Algebra I / Geometry / Algebra II strands. Do not leave the transcript as "Math Year 1 / Math Year 2."

Here is an illustration, labelled as such, not a reported family:

> 4 September 2026. Precalculus. Task: locate $5\pi/6$ on the unit circle; give exact sine and cosine; one sentence on sign. Student marked the reference angle, wrote = 1/2, = -32, checked the quadrant. Missed a

[76] NCAA Eligibility Center, *Home School Toolkit* 2025–26 (January 2026 PDF opened 5 September 2026): sample academic record uses Algebra I, Geometry, Algebra II, Precalculus. Transcript required fields include ninth-grade start date (mm/dd/yyyy), credits in allowed increments, no course more than 1.0, grading scale, graduation date, administrator signature.

[77] Algebra I before Geometry is typical kitchen practice and matches the NCAA sample; it is not a federal statute. Pennsylvania names algebra and geometry as secondary shall-include content without ordering them by year. New York names mathematics units without naming Algebra I. Integrated school pathways (Core-Plus; Cognitive Tutor Integrated Math) exist as school objects — export stranger-readable titles or a course description.

> later item with a negative angle; rebuilt from the unit circle; wrote the check.

That is a record a parent, a Pennsylvania evaluator, a New York quarterly narrative, or an NCAA Core-Course Worksheet can actually use. It is not a credit factory. It is not a diploma.

The law is a pattern, not one form

U.S. homeschooling is state law. There is no federal math office. Dates and subject lists change. Read your current department page and the statute it cites. A color-coded chart on a membership site is not your statute. Five official-page examples show the range. Math is the usual required or assumed high-school object.[78]

Texas. The Agency does not regulate or accredit home-school programs. *Leeper*, restated on TEA's Alternative Schooling page, requires a bona fide written curriculum including **math**. The state does not award a homeschool diploma. For colleges and NCAA, the parent builds a transcript with stranger-readable titles. Transfer is treated like an unaccredited private school; a district may test for placement. "We did Saxon Advanced Math" is not, by itself, a public-school credit.

North Carolina. Notice of Intent. Nine calendar months. Each year, a nationally standardized test including **mathematics**. The home school, not the State, issues any diploma. A five-hour day is a recommendation, not a math-minutes table. Career and College Promise on the DNPE page is **North Carolina** dual enrollment with readiness gates — state-variant, not a national right. The annual battery is compliance, not this book's teaching check.

Pennsylvania. Secondary **shall include mathematics, to include general mathematics, algebra and geometry**; trigonometry **may**. Graduation includes **three years of mathematics**. Portfolio tests in grades 3, 5, and 8 include **mathematics**. 990 secondary hours are whole-program hours. A supervisor (or approved-organization) diploma is Pennsylvania's high-school object.

Virginia. Notice by 15 August, a parent-written **list of subjects**, and evidence of progress by 1 August (fourth stanine, or SAT/ACT/PSAT equivalent, or evalu-

[78] Five-state statute pages as in the body (TEA *Leeper*; NC DNPE; PA 24 P.S. § 13-1327.1; NY 8 NYCRR 100.10; VA § 22.1-254.1). This chapter is not legal advice. The parent reads their own statute.

ation / college transcript). Mathematics is the ordinary word to write on the list. Evidence of progress is compliance, not a college transcript. A publisher math placement PDF does not, by itself, satisfy the fourth-stanine object.

New York. Letter of intent, IHIP, quarterly reports, and an annual assessment. Grades **9–12** require **mathematics (two units)**; 990 hours for grades 7–12 are whole-program hours. Put the work under mathematics in the quarterlies. Home instruction does not yield a local or Regents diploma. For NCAA proof of graduation, the Toolkit notes New York (and Hawaii) paths that are not a local diploma — open the Toolkit.[79]

If you live in none of those states, open your department page. Pennsylvania's three years and New York's two units are floors — colleges and the NCAA often want **more**. Compliance and admissions are different readers.

What a short check is

Four objects, not a battery.

1. **A dated notebook.** Date, skill named, what the student built or wrote, one sentence of what happened — under **Algebra I**, **Geometry**, **Algebra II**, **Precalculus**, **Calculus**, or **Statistics** if a title is needed. Mistakes visible. Best samples: a corrected free-response with a reason; a marked geometry diagram; a function-family table without a regression button; a unit-circle location with sign; a rate table with units; a sample-vs-population sticky.

2. **An exit ticket.** Three to five items at the end of a math hour or week, one of them not today's new skill. Right, or wrong-with-a-reason you can use tomorrow, is enough. Not a percentage identity. Include at least one item that requires written reasoning so the notebook stays richer than an app score.

3. **One diagnostic item.** A single-page skill probe — solve a linear and a quadratic with justification; write a short proof with reasons; sort a table into linear / quadratic / exponential; place an angle on the unit circle; read an average rate with units; name sample vs population — used to place or

[79] NCAA Home School Toolkit: New York and Hawaii do not recognize a homeschool diploma for proof of graduation in the Toolkit's sense — district/state letter or equivalency path. Point to the Toolkit.

to re-place midyear. Not a forty-item diagnostic. Keep one wrong answer you can reteach. That is the file of the habit.

4. **Optional publisher placement** (the organization sells the test). AoPS "Are you ready?" / "Do you need this?"; Teaching Textbooks trial lessons; CTC-Math topic diagnostics; Saxon skill-level tests; publisher readiness lists; OpenStax chapter quizzes as a kitchen restart. Use to start or restart a purchased program. IES does not certify them as randomised trials. They place by skill; they do not license a national percentile; a publisher math placement PDF generally is not North Carolina's annual battery or Virginia's fourth-stanine object.

Date the work, keep one wrong answer, leave the pencil in the student's hand while they try.[80]

What a short check is not

NAEP. A percentile promise. A homemade high-stakes battery. Terra Nova as the kitchen model. MAP as an identity. CLEP as a substitute for a year's notebook. AP as the only score that counts. SAT Subject Tests — discontinued, last international sittings June 2021. A convenience-sample homeschool scoreboard. This book does not reprint that fight.

Dual enrollment, AP, and CLEP — options, not the spine

Dual enrollment is a real release valve when the parent is the ceiling — Algebra II through Calculus at a community college, with an official college transcript admissions offices already know how to read. State variance is large. North Carolina's Career and College Promise is one opened example with readiness gates. Other states have their own rules. Check your state and the college's dual-enrollment office. Do not invent a national right. On a homeschool transcript, designate the

[80] Erin A. Maloney et al., "Intergenerational Effects of Parents' Math Anxiety on Children's Math Achievement and Anxiety," *Psychological Science* 26 (2015): 1480–1488. School homework-help study (grades 1–2), not a high-school homeschool trial. Cited for the kitchen move: leave the pencil with the student.

course as a college course with the college's name and city or state; for NCAA, submit the official college transcript separately.[81]

AP Calculus AB, AP Calculus BC, and AP Statistics (and AP Precalculus as a neighboring College Board course) are optional external exams. Homeschoolers sit at a participating school with an AP coordinator; parents cannot order exams alone. Opened fee language for the 2027 sitting cycle: about **$99** in the U.S., Canada, and DoDEA; about **$129** outside — confirm with the coordinator.[82] An AP score is a signal, not the only path that counts, and not this book's teaching spine.

CLEP College Algebra, Precalculus, and Calculus are optional credit-by-exam sittings where a college's policy accepts them. Opened exam fees sit near **$97** each — confirm on the current College Board page before you register.[83] CLEP can earn college credit. **CLEP is not an NCAA core course.** Audited and credit-by-exam courses are not NCAA-approved core courses. Say that once, clearly, if athletics are in view.

The spine of this book is understanding Algebra I through Statistics well enough to hear a wrong answer and to export a stranger-readable record. Dual enrollment, AP, and CLEP sit beside that spine when they fit.

If athletics are in view — point to the Toolkit

NCAA eligibility is a private association's rules, not law, but it binds athletic scholarships. Source hierarchy for this book: the **NCAA Home School Toolkit**, Core-Course Worksheet instructions, and ncaa.org Division pages. Not blogs. Not Facebook groups.[84]

[81]Dual enrollment: state variance; NC CCP opened on DNPE as example. NCAA Toolkit: designate college courses on the homeschool transcript; submit official college transcript; Core-Course Worksheet not required for the college course itself. PFI 2016 grades 9–12 college cell 4! (interpret with caution).

[82]College Board AP Students fees page (opened 5 September 2026): 2027 exams — $99 U.S./Canada/DoDEA; $129 outside. Homeschoolers need a participating school's AP coordinator. Confirm current fees.

[83]College Board CLEP exam topic pages (opened 5 September 2026): College Algebra, Precalculus, Calculus — register fee $97.00 displayed. CLEP is not an NCAA core course (Toolkit: audited, CLEP, and credit-by-exam courses are not NCAA-approved core courses).

[84]NCAA Home School Toolkit 2025–26 and ncaa.org Division pages (opened 5 September 2026): math = Algebra I or higher; DI 3 years math / 2.3 core GPA among 16 core courses; DII 2

Math that counts as a core course is **Algebra I or higher**. Division I wants **three** years of math among sixteen core courses and a minimum **2.3** core GPA. Division II wants **two** years and a minimum **2.2** core GPA — confirm the current Division pages at publication. The homeschool packet includes a transcript with a ninth-grade start date (month/day/year), credits in allowed increments, no course more than 1.0 credit, an Administrator and Accordance Statement, **one Core-Course Worksheet per core course**, and proof of graduation. Dual-enrollment college courses need transcript designation and an official college transcript; a Core-Course Worksheet is not required for the college course itself. Evaluation happens after an NCAA school places the student on an institutional request list. Proof of graduation may be a diploma, a transcript with a graduation date, or — in New York and Hawaii per the Toolkit — a district or state letter or equivalency path. State compliance is not automatically NCAA proof. Open the Toolkit and copy what it says rather than a blog's rules.

Before you title a year Calculus

Check — at minimum — that Algebra II function families and Precalculus trig / unit-circle work are secure enough that rate and accumulation will mean something. Publisher readiness lists and AoPS Are-you-ready pretests are one articulation of that gate. Neither is a state law. A seventeen-year-old who fails the gate does **Precalculus** or finishes **Algebra II** honestly. A fifteen-year-old who clears the gate may begin calculus meaning with supervision — skill and motivation, not a race. Birthday is a weak instrument. Skill is the instrument.[85]

Re-placement midyear is allowed. Spiral programs and mastery programs both produce students who look "behind" or "ahead" of a birthday grade. The short check's job is to notice. Pride is not a placement instrument. Statistics can sit beside or after Algebra II; it need not wait for Calculus.

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years math / 2.2 core GPA; packet checklist as in the Toolkit. Confirm Division II language and any test-score policy at publication. Do not invent rules from blogs.

[85] Calculus by readiness, not birthday: NMAP Finding 15 continuity; Critical Foundations if unfinished. AoPS Are-you-ready / Do-you-need-this pretests — vendor tools, not state law. IES / WWC does not certify publisher placement tests as randomised trials.

If they transfer or apply

Transfer and admissions are local placement problems. A brand in the title column will not be read as Algebra II. Another reason the title should already look like Algebra I, Geometry, Algebra II, Precalculus, Calculus, or Statistics.

A receiving counselor who sees "Emerging Adults Math I, A" has been given a slogan. A counselor who sees "Algebra II — dated notebook; function families from tables without a regression button; exponential growth context sentence; check by graph" has been given a year they can place.

Many college pages ask for about three years of mathematics through Algebra II as a floor; selective STEM programs often want Precalculus or Calculus plus an external signal — confirm the destination school's current page.

Place by skill, not birthday

Can the student solve a linear equation by the same operation on both sides and say why? If not, they are still in Algebra I repair, whatever their age. Can they write a short proof with reasons in a chosen format? If not, they are still in Geometry. Can they name a table as linear, quadratic, or exponential without a regression button? If not, they are still in Algebra II. Can they place an angle on the unit circle and explain the sign? If not, they are still in Precalculus. Can they say what an average rate is becoming as the interval shrinks — with units? If not, they are not ready for calculus meaning yet. High-sounding *titles* cannot skip those moves. An eighteen-year-old may still need linear/quadratic structure. A fifteen-year-old may already clear a calculus gate. Skill first.[86]

What to keep in the file

1. A dated notebook: date, skill named, what the student did, one wrong answer you will reteach. Under **Algebra I**, **Geometry**, **Algebra II**, **Precalculus**, **Calculus**, or **Statistics**.

[86] Place by skill: NMAP Finding 15. PFI 2016 still-arithmetic 31 percent *that year* in grades 9–12. PA/NY 990 hours are whole-program; NC five-hour day is a recommendation. NAEP Grade 12 2024 (~147 average; 22% ≥ Proficient; 45% below Basic) is school population, not a homeschool census.

2. Exit tickets from the week, not a crate of every worksheet. App gradebooks are useful parent reports; they are not a substitute for a work sample a Pennsylvania evaluator can hold.
3. One diagnostic item, labelled — including a readiness gate when Calculus or Algebra II is on the table.
4. Optional publisher placement, if you used one to start or restart a program — as a placement, not as a yearly IQ.
5. Optional external layers: dual-enrollment college transcript; AP score report; CLEP score where a college accepts it — never as a fake substitute for a blank year.
6. State objects that actually apply: a North Carolina annual mathematics-inclusive battery if you are in North Carolina; a Pennsylvania secondary mathematics portfolio sample if you are in Pennsylvania; a New York quarterly under mathematics if you are in New York; a Virginia list that includes Mathematics if you write the list; a Texas parent transcript when a stranger asks. Compliance stays in its lane.
7. If athletics are in view: Core-Course Worksheets and the Toolkit checklist — separate from state compliance.

Life of the habit, as the file you keep, lives here: date the work; name the skill; keep one wrong answer you can reteach. Revisit next Tuesday is a move in the math hour and in each chapter's done-enough checklist. The notebook is how Tuesday leaves a trace.

A path through high-school mathematics is the promise. A diploma is not. A percentile is not. An AP score is not the only path that counts.

Chapter 8

Resources

This chapter names programs families actually meet at ages 15–18. It does not rank them. It does not sell them. It is a fit list so you can match an object to the table you already have: how much of the teaching you will carry, whether a kitchen can use it tonight, what it costs in money and in minutes, and whether the student will still solve with structure, write a reason, sort a function family, place an angle on the unit circle, and — when ready — talk rate and accumulation with units.

A "grade 11 Algebra II workbook" is a publisher's scope, not a legal grade. Placement is by skill. The record, as Chapter 7 said, still says **Algebra I**, **Geometry**, **Algebra II**, **Precalculus**, **Calculus**, or **Statistics** — only if the year's work was that course. Combining two honest objects does not mint a new title. Kitchen function-family cards plus library book plus one lesson from a purchased program is still Algebra II.

Independent trials of these programs *in homeschool* remain scarce. What Works Clearinghouse ratings, where they exist for secondary products, are school-side — not a kitchen bake-off. Choose by fit: parent load, style, budget, age or grade label, and whether the student will generate the work.

This week, name *why* tonight's object fits — usually a scrap-paper solved problem and a dated notebook under a stranger-readable title. Today the student tries that problem.

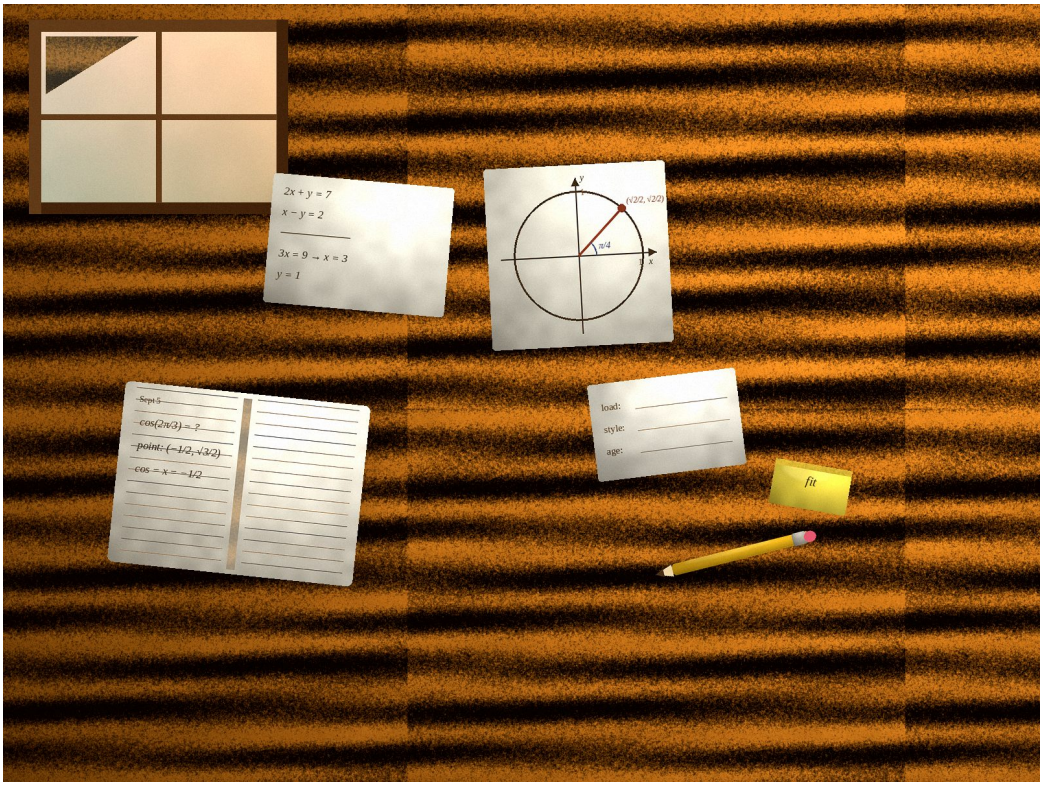

Figure 11: A scrap-paper solved problem, a unit-circle sketch, a dated notebook, a blank fit card (load / style / age label). No logos. No portraits.

How to read a program

Four questions do the work.

Parent load. Are you asking one good question on a solved problem the student already attempted? Sitting with a scripted lesson every day? Learning teaching notes well enough to hear letter-moving with no structure? Booking a video or an outside teacher as a second mouth?

Style. Traditional college-prep text? Spiral mixed practice? Problem-solving depth? Short videos for several siblings? Geometry with reasons and a video coach? Narrative story-math? Free OpenStax chapter plus your questions?

Cost shape. Nothing tonight. A book you already own. A free OpenStax or Khan

map. A print set or online course whose dollars sat on a page this access day and will stale. Fees this book did not verify. Confirm a current price on the page before you buy.

Age or grade label. A fifteen-year-old is not automatically Algebra II. An eighteen-year-old may still need Algebra I repair. Birthday is a weak prior. Publisher age bands and grade labels are starting hints; skill gates decide. Calculus is for ready students — not every eighteenth birthday.

Kitchen table, nothing spent

A dated notebook. Scrap paper for a solved problem. A marked geometry diagram. Function-family sort cards from a table. A unit circle drawn at the dinner table. A rate table with units. A sample-vs-population sticky. A car-loan amortization row as a math object — not a sermon about debt. A bridge angle or map bearing you were already near. A paycheck percent from a stub you already had. A library text. OpenStax Algebra and Trigonometry or Calculus as a free download. Khan Academy practice maps. That is tonight.

Fit. One student, low cash cost, parent load moderate to high if you invent the sequence — moderate if a free course map or a borrowed text carries scope. In 2016, tutors, co-ops, and online courses already shared the high-school load for many families (tutor any instruction 24 percent; co-op 37 percent; online courses 30 percent in grades 9–12).[87] A video as a second mouth is ordinary when you are the ceiling — not a failure of the kitchen.

Transcript. Algebra I / Geometry / Algebra II / Precalculus / Calculus / Statistics by content. Never Emerging Adults Math I. Never "Consumer Math 12." Never Outdoor Math 1.

The teaching chapters of this book are written for this object first. A purchased program is a fit, not a requirement.

[87] NCES 2020-001, Table 3: grades 9–12 — mother as main provider 70 percent; any tutor 24; any co-op 37. Table 7: courses online 30 percent in 9–12. The 2023 First Look did not republish these tables. Access date for URLs in these notes: 5 September 2026.

OpenStax and Khan Academy — free first after the kitchen

OpenStax. Free college-level math textbooks — Algebra and Trigonometry and related titles in the OpenStax catalog, plus Calculus volumes.[88] Fit: $0 text for Algebra II / Precalculus / College Algebra grain, and for calculus meaning when the student is ready. Parent still builds lessons, exit tickets, and assessments — or pairs the text with dual enrollment. Good bridge text when a community-college syllabus already names OpenStax. Confirm the current catalog page before you print.

Khan Academy. Free K–12 math courses and practice maps.[89] Fit: $0 practice; parent still owns placement and transcript titles. Do not treat Khan alone as a complete program of record unless you date the notebook and title the year honestly. COPPA does not apply at ages 15–18; the pedagogical lock from the math hour still holds — parent co-holds or can audit the account; unaided first; no secret friend during the attempt. Transcript: never "Khan" as the credit.

Transcript for both. Algebra I / Geometry / Algebra II / Precalculus / Calculus / Statistics by content.

Art of Problem Solving

Deep secondary curriculum: Introduction to Algebra → Introduction to Geometry → Intermediate Algebra → Precalculus → Calculus (and related Scholars offerings).[90] Parent load is moderate to high: hard problems; the parent may not spot-check quickly; an online class shifts load to an instructor. Style: problem-solving,

[88] OpenStax catalog: Algebra and Trigonometry and Calculus titles among free college-level math texts. Parent still owns lessons and assessment unless paired with dual enrollment. Confirm current catalog page.

[89] Khan Academy: free K–12 math courses and practice maps (About / course-map pattern). Organization efficacy copy is not a homeschool RCT opened as a PDF for this chapter. COPPA does not apply at 15–18; pedagogical co-hold / audit and unaided-first rules remain (see the math hour).

[90] Art of Problem Solving store and recommendations pages, accessed 5 September 2026. Sequence: Introduction to Algebra → Introduction to Geometry → Intermediate Algebra → Precalculus → Calculus. Are-you-ready / Do-you-need-this diagnostics. Calculus offering described by vendor as College Board–approved AP Calculus BC class.

proof-leaning, contest-adjacent. “Are you ready?” and “Do you need this?” diagnostics place by content experience, not by age restriction. Vendor copy describes a Calculus offering as College Board–approved AP Calculus BC for grades 9–12 and beyond a standard BC curriculum — that is the organization’s description, not a kitchen RCT.

List prices and online tuition change. Confirm on the current store and class pages before you buy.[91]

Fit. Problem-solving appetite; pretest placement; coach or online seat. High rigor. Not the default kitchen hour for every family.

Transcript. Algebra I / Geometry / Algebra II / Precalculus / Calculus by content — never “AoPS Math 9.” If Algebra I was already granted and AoPS is for depth, say enrichment in the description; do not mint a silent duplicate credit. Contest prep (AMC/AIME) is enrichment, not a transcript credit unless you honestly frame a separate elective — and even then, strangers want the core titles first.

Saxon Advanced Math and Calculus

Incremental development plus continual review. Secondary courses named in school-side reviews include Algebra I, Geometry, Algebra II, and Advanced Math; Calculus sits in the homeschool conversation as a next spiral course.[92] Parent load is moderate to high early in a course (lesson plus mixed practice), lower when the student is independent. Style: spiral mixed practice as the point. Homeschool kit SKUs and current list prices — confirm on the current vendor page before you buy.[93]

School-side, labelled school, not a kitchen trial: the What Works Clearinghouse Saxon Algebra I report (May 2016, Secondary Mathematics) found **no discernible effects** on algebra. Under that review there was no evidence for Saxon Geometry,

[91] AoPS list prices and online tuition: confirm on the current store and class pages. This chapter omits stale list prices that were not locked as opened durable figures for every SKU.

[92] Saxon incremental development + continual review: WWC program description language. Secondary courses named in WWC Secondary Mathematics reviews: Algebra I, Geometry, Algebra II, Advanced Math. Calculus in homeschool catalogue conversation — confirm current kit naming on the vendor page.

[93] Saxon / HMH homeschool kit dollars: confirm current vendor price. School-side ESSA language from the publisher is not a homeschool RCT.

Algebra II, or Advanced Math.[94] That is school. It is not a kitchen RCT. Do not mash it with the primary-grades Saxon mixed-effects rating from a different protocol — different band, different report.

For Advanced Math, confirm whether **Precalculus** or **Algebra II / Trig** is the honest transcript title for *your* year's scope. The materials line may say Saxon Advanced Math. The credit line stays stranger-readable.

Fit. A family that wants spiral mixed practice and can live with review that returns last week's skill.

Transcript. Algebra I / Geometry / Algebra II / Precalculus / Calculus by content. Never "Saxon Advanced Math" alone as the credit.

Foerster

Classic college-prep Algebra, Precalculus, and Calculus texts in the homeschool grapevine (Pearson lineage).[95] Style: traditional college-prep text — worked examples, problem sets, structure a stranger expects on a materials line. Detailed current-edition feature lists were thin on the pages opened for this book; confirm the edition and teacher supports on a primary product page before you rely on a specific feature. Parent load is moderate to high if you teach from the text; lower if a co-op or tutor carries the first explanation.

Fit. A family that wants a traditional college-prep book and can hear letter-moving with no structure as a missing-meaning move.

Transcript. Algebra I / Algebra II / Precalculus / Calculus. Never "Foerster" as the credit.

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[94] WWC Saxon Algebra I secondary report, May 2016: no discernible effects on algebra; no evidence for Geometry / Algebra II / Advanced Math under that review. School-side — separate from the primary-grades Saxon report.

[95] Foerster: traditional college-prep Algebra / Precalculus / Calculus texts (Pearson lineage) in homeschool use. Detailed feature cards were thin on pages opened 5 September 2026 — confirm edition/features on a primary product page. Style labelled; features not invented.

Jacobs Geometry

Harold Jacobs' Geometry, often packaged with AskDrCallahan video or online support: conversational, real-life application approach; reasons and understanding before formula-sheet panic.[96] Parent load is moderate when the video carries the first explanation; you still need to hear a proof with no bridge. An opened AskDrCallahan online-course component listed lifetime access near **$149** on the access day for this book — a rare opened price; confirm before you buy, because vendor prices stale.[97] The same site also supports Algebra, Algebra II with Trig, and Calculus packaging — confirm current offerings.

Fit. Geometry with understanding; DVD or online teacher substitute for the parent; formats optional, reasons required (as Chapter 2 teaches).

Transcript. Geometry. Never "Jacobs" or "AskDrCallahan" as the credit.

Teaching Textbooks, CTCMath, Derek Owens, and Thinkwell

Teaching Textbooks. Homeschool math app spanning elementary through high school; vendor claims include graded problems, step-by-step audiovisual solutions, tutor helpline, automatic grading, mobile or offline work for a short stretch of lessons, free trial.[98] Parent load is **low** — the vendor's main pitch. Homepage views opened for this book did not display a clean public price table; confirm current price. Style: screen, often described in catalogues as a gentler pace — relevant for later transfer; keep a parallel skill diagnostic in the notebook so placement is not a surprise. App reports are not a substitute for a work sample. Transcript: Algebra I / Geometry / Algebra II / Precalculus by content — never "TT" as the credit.

CTCMath. K–12 video tutorials, interactive questions, worksheets and solutions,

[96] Jacobs Geometry via AskDrCallahan packaging: vendor pages opened 5 September 2026 — conversational, real-life application approach; online course packaging.

[97] AskDrCallahan online course component lifetime access listed at $149.00 on the opened page (5 September 2026). Vendor prices at access; they will stale. Confirm before you buy.

[98] teachingtextbooks.com homepage, accessed 5 September 2026: app features and free trial described; no clean public price table on the fetched view that day. Confirm current price.

reports, topic diagnostics; free trial.[99] Parent load is **low** if the student can watch and work; parent reads reports. Modular topics help catch-up and keep-up; they make year titles harder — the parent must still write a coherent title based on dominant work. Confirm current price on the page. Transcript: not "CTCMath Grade 10" as the credit.

Derek Owens (Lucid Education). Online and live classes: Algebra 1, Geometry, Algebra 2, Precalculus, **AP Calculus**, and neighbors; online students may begin any time.[100] Fit: teacher-of-record outside the kitchen — useful when worksheets honesty and an outside grade matter. Tuition is the cost shape; confirm current enrollment pages. Transcript: still Algebra I / Geometry / Algebra II / Precalculus / Calculus; the outside instructor helps the worksheet and the gradebook; you still own the stranger-readable title unless the provider issues its own transcript.

Thinkwell. Video-lecture homeschool math by catalogue reputation.[101] Detailed current course lists and prices were thin in the extracts opened for this book — confirm features and price on the live page before you buy. Fit: video as first mouth; parent still dates the notebook.

Shared transcript rule. Low parent load on a screen is a fit, not a pass on work samples. Title by content.

Life of Fred — style-labelled

Stanley Schmidt's narrative / story-style math.[102] Label it honestly as narrative style from catalogue consensus. Detailed scope-and-sequence claims beyond what a thin author page shows are not invented here — confirm the current catalog before you plan a full high-school arc from the stories alone. Fit: a student who thrives on story and still needs a parallel skill check so Algebra II does not be-

[99] ctcmath.com homepage, accessed 5 September 2026: K–12 video, interactive questions, reports, free trial; list price not displayed that day. Confirm current price.

[100] Derek Owens / Lucid Education pages, accessed 5 September 2026: online and live classes including Algebra 1, Geometry, Algebra 2, Precalculus, AP Calculus; online students may begin any time. Confirm current tuition and enrollment.

[101] Thinkwell: video-lecture homeschool math by catalogue reputation; detailed 2026 course list and prices thin in extracts opened this access. Confirm live page before detailed claims.

[102] Life of Fred / Stanley Schmidt site: narrative / story-style math labelled from catalogue consensus; thin curriculum detail on the opened author page this access. Do not invent chapter lists. Confirm current catalog.

come “we finished the novel.” Transcript: Algebra I / Geometry / Algebra II / Precalculus / Calculus by content if the year’s work maps — never “Life of Fred” as the credit.

School programs you will see named — WWC secondary, labelled school

These are school-side What Works Clearinghouse ratings, labelled school, copied not rounded, not a reason to crown a franchise at the table.[103]

Saxon Algebra I (May 2016, Secondary Mathematics) — **no discernible effects** on algebra. No evidence under that review for Saxon Geometry, Algebra II, or Advanced Math.

Cognitive Tutor® Algebra I (June 2016 revised) — **mixed effects** on algebra; no discernible effects on general mathematics. **Cognitive Tutor® Geometry** — **potentially negative effects** on geometry.

UCSMP Algebra (May 2016) — **potentially positive effects** on general mathematics and algebra. No evidence under that review for UCSMP Geometry, Advanced Algebra, Functions/Statistics/Trigonometry, or Precalculus.

Core-Plus Mathematics (September 2010, High School Math protocol) — **potentially positive effects** on mathematics achievement for high-school students. Integrated four-year sequence; small evidence base. If a kitchen uses an integrated text, export stranger-readable titles or a course description — do not leave “Math Year 2” on the transcript.

Singapore Math, named so the sentence stays attached: WWC December 2015, Primary Mathematics protocol — **no studies meeting group design standards**; the WWC is unable to draw conclusions about effectiveness or ineffectiveness.[104] That is a primary protocol finding. It is not a secondary Singapore effectiveness

[103] WWC reports as in the body (Saxon Alg I May 2016; Cognitive Tutor June 2016 rev.; UCSMP Algebra May 2016; Core-Plus Sept 2010). All school-side. None is a parent-at-home RCT.

[104] What Works Clearinghouse, *Singapore Math*® intervention report, Primary Mathematics protocol, K–8, December 2015: no studies within scope meet WWC group design standards; WWC unable to draw conclusions about effectiveness or ineffectiveness. Copy this finding whenever Singapore Math is named. Do not invent a secondary Singapore effectiveness claim.

claim for ages 15–18. Bar models and strip diagrams remain usable representations. That is not a curriculum endorsement.

None of these is a parent-at-home randomised trial. None is a kitchen winner. A school rating is useful context. It is not a promise that every home will see the same result.

Extra help, age-labelled — school maths, not a kitchen promise

Named English *maths* trials only, copied honestly, labelled by year group / age, as an analogy for coherent focus — not as a ranking of kitchen curricula, and not as "+1 month at your table."

Ark Mathematics Mastery **Secondary**, Year 7 (~age **11–12**): **+1 month**, **4 of 5 padlocks** — younger than this book's core band.[105] Maths in Context (KS4): **0 months**.[106] 5Rs (ages **16–19**): first trial reported no robust impact.[107] Portable moral only: coherent focus can move attainment modestly in school trials — not a kitchen promise. Philosophy for Children is **not** a mathematics program.[108]

When you are the ceiling, a tutor, co-op seat, dual enrollment, or video as a second mouth is ordinary. Unless the outside provider issues a transcript, you date the notebook under a stranger-readable title.

Placement tests are tools, not percentiles

AoPS Are-you-ready / Do-you-need-this; Teaching Textbooks trial lessons; CTC-Math topic diagnostics; Saxon skill-level tests; publisher readiness lists; OpenStax chapter quizzes as a kitchen restart. IES does not certify them as randomised trials. They place by skill. They do not license NAEP. They do not write Emerging Adults Math I. Use one to start a purchased program or to restart after a gap, as Chapter 7 said.

[105] EEF Ark Mathematics Mastery Secondary: Year 7 (~11–12), +1 month, 4/5 padlock — school; younger than this band.

[106] EEF Maths in Context (KS4): 0 months.

[107] EEF 5Rs (16–19): first trial, no robust impact.

[108] EEF Philosophy for Children is not a mathematics program.

A student of sixteen may place into three different "grades" on three tests the same afternoon. That is three grains of "grade," not a contradiction. Start where the student can succeed, then add. Algebra I before Geometry is typical, not statute. Calculus by readiness, not birthday.

A short chooser, not a ranking

- Nothing spent tonight: dated notebook, scrap-paper solved problem, library text, OpenStax, Khan — title by content.
- Free college-text grain: OpenStax Algebra and Trigonometry / Calculus; you still write exit tickets. Free practice maps: Khan — you still write the record.
- Problem-solving depth: AoPS Intro → Geometry → Intermediate Algebra → Precalculus → Calculus; Are-you-ready first. Never AoPS as the credit.
- Spiral mixed practice: Saxon through Advanced Math / Calculus by skill. WWC Saxon Algebra I: no discernible effects (school). Confirm whether Advanced Math exports as Precalculus or Algebra II/Trig for *your* year.
- Traditional college-prep text: Foerster — confirm current edition. Geometry with a video coach: Jacobs + AskDrCallahan (opened online-course component ~$149 on one access day — confirm).
- Low parent load on a screen: Teaching Textbooks or CTCMath — confirm price; keep paper samples for portfolio states.
- Outside teacher of record: Derek Owens — tuition; title still stranger-readable unless they issue the transcript. Video lecture: Thinkwell — confirm live features. Narrative style: Life of Fred — keep a parallel skill diagnostic.
- School catalog names Cognitive Tutor, UCSMP, or Core-Plus: copy the WWC school ratings above; they are not a home promise. Singapore secondary edges: WWC December 2015 Primary Mathematics — no studies meet; that finding is not a secondary win.

None of those bullets is "best." Fit is whether you can hear a wrong answer in that program's pictures, and whether the student will generate the work. When the student has unfinished linear/quadratic structure, go back, whatever the calendar says. When you are the ceiling, buy a tutor, a co-op seat, dual enrollment, or a video as a second mouth. When the destination is a stranger with a page in hand, title the year Algebra I, Geometry, Algebra II, Precalculus, Calculus, or Statistics

— the mathematics, not the franchise.

A path through high-school mathematics is the promise. A diploma is not. A percentile is not. A catalogue star is not a finding.[109]

[109] No homeschool-curriculum RCT comparing these named programs was opened for this book. Catalogue reviews are not trials. Confirm current prices; rare opened figure: AskDrCallahan online-course component ~$149 at access.

A Note on Sources

Studies named in the chapters are listed in Notes, in one series at the back. That is where the full citations live, so the teaching pages can stay a teaching voice.

Some items the research behind this book did not open stay out of the teaching voice. I did not invent a coefficient, a statute, or a product feature to fill a hole. If a study is in the chapter, it is in the notes. If we could not open it, it is not used as a finding here.

Program hours, credit claims, and product descriptions in the resources chapter are taken from the companies’ own pages. Access date 5 September 2026.